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INVARIANCE PROPERTIES OF TIME-VARYING LINEAR SYSTEMS

Octavian Pastravanu and Mihaela Matcovschi

Department of Automatic Control and Applied Informatics
Technical University "Gh. Asachi" of Iasi
Blvd. Mangeron 53A, Iasi 700050, Romania
Phone / Fax: +40-232-230751; E-mail: {opastrav, mhanako}@ac.tuiasi.ro

Abstract: This paper formulates and proves two types of necessary and sufficient conditions for the characterization of positively (flow) invariant sets with respect to the state-space trajectories of the time-varying (non-autonomous) linear systems in both continuous- and discrete-time case. These conditions are expressed in terms of inequalities involving the matrix function that defines the system dynamics and a constant matrix that defines the shape of the invariant set. The first type of results refers to contractive invariant sets which decrease exponentially, and the second one considers invariant sets that remain constant. Our approach to non-autonomous systems accommodates, as particular cases, the elements of the invariant set analysis already elaborated for autonomous systems.

Keywords: time-invariant (autonomous) linear systems, time-varying (non-autonomous) linear systems, constant invariant sets, contractive invariant sets.

1. INTRODUCTION

The exploration of the invariance properties for the free response of dynamical systems has presented a great interest for many research groups, starting with the mid eighties. This interest is reflected by works such as (Pavel, 1984), (Voicu, 1984), (Voicu, 1984), (Kiendl *et al*, 1992), (Boyd *et al*, 1994), (Michel and Wang, 1995), (Hmamed and Benzaouia, 1997), (Blanchini, 1999), (Kaszkurewicz and A. Bhaya, 2000), (Kaczorek, 2002), (Gruyitch *et al*, 2004), (Pastravanu and Voicu, 2005), (Pastravanu and Voicu, 2006) and numerous other papers cited therein.

This interest was motivated by analysis and synthesis objectives, oriented towards the preservation of the trajectories inside certain sets defined in the state space. The literature shows that most approaches focused on the dynamics of time-invariant (autonomous) linear systems, by considering polyhedral sets invariant with respect to the trajectories. The attention paid to this class of systems is motivated by the possibility to formulate, in terms of system matrices, easy to handle algebraic conditions which characterize the invariance properties.

Belonging to this research trend, but aiming to enlarge the insight into the invariance problem, paper (Kiendl *et al*, 1992) refers to invariant sets defined by Hölder vector p-norms, $1 \le p \le \infty$. Thus, for a given *continuous-time* (CT) invariant (autonomous) linear system

$$\dot{x}(t) = A\,x(t),\ x(t_0) = x_0 \in \mathbf{R}^n ,$$
$$t, t_0 \in \mathbf{R}_+ ,\ t \ge t_0 ,\ A \in \mathbf{R}^{n \times n} , \qquad (1)$$

let us consider the solutions $G \in \mathbf{R}^{n \times n}$, $\mathrm{rank}(G) = n$, of the inequality

$$m_p(GAG^{-1}) < 0 , \qquad (2)$$

where

$$m_p(GAG^{-1}) = \lim_{\theta \downarrow 0} \frac{\| I + \theta GAG^{-1} \|_p - 1}{\theta} \tag{3}$$

is a measure of the matrix GAG^{-1}. According to (Kiendl *et al*, 1992), the sets built with the solutions G of inequality (2) and having the form

$$X_{p,G}^c = \left\{ x \in \mathbf{R}^n \,\middle|\, \| Gx \|_p \le c \right\}, \; c > 0 , \tag{4}$$

are invariant with respect to the trajectories of the autonomous CT linear system (1), i.e. any trajectory that starts in the set will never leave it.

Correspondingly, for a given *discrete-time* (DT) invariant linear system

$$\begin{gathered} x(t+1) = A\,x(t),\; x(t_0) = x_0 \in \mathbf{R}^n , \\ t, t_0 \in \mathbf{Z}_+ ,\; t \ge t_0 ,\; A \in \mathbf{R}^{n \times n} , \end{gathered} \tag{5}$$

let us consider the solutions $G \in \mathbf{R}^{n \times n}$, $\mathrm{rank}(G) = n$, of the inequality

$$\| GAG^{-1} \|_p < 1 , \tag{6}$$

where

$$\begin{aligned} \| GAG^{-1} \|_p &= \sup_{y \in \mathbf{R}^n ,\, y \ne 0} \frac{\| GAG^{-1} y \|_p}{\| y \|_p} = \\ &= \max_{y \in \mathbf{R}^n ,\, \|y\|_p = 1} \| GAG^{-1} y \|_p \end{aligned} \tag{7}$$

is the norm of matrix GAG^{-1}, induced by the Hölder vector p-norm $\| \; \|_p$. Similarly to the CT case, (Kiendl *et al*, 1992) shows that the sets of form (4) built with the solutions G of inequality (6) are invariant with respect to the trajectories of the autonomous DT linear system (5).

The objective of the current paper consists in proving that similar characterizations of the invariant sets can be formulated for time-varying (non-autonomous) linear systems in CT and DT. Despite the formal analogy of the results, our work is not a trivial extension of the autonomous case addressed by (Kiendl *et al*, 1992) [commented in (Hmamed *et al*, 1994) and (Loskot *et al*, 1998)] since the mathematical tools for the qualitative analysis of time-varying systems are different (as well-known from the stability theory). For this class of systems we commence our study with the general problem of contractive and invariant sets, and, subsequently, the constant invariant sets are treated as a particular case. Our results are formulated as necessary and sufficient conditions for set invariance. For simplicity reasons, the whole construction is detailed in Section 2 for CT dynamics, and only sketched in Section 3 for the DT case, since the latest is merely a parallel *mutatis mutandis* approach to the first one. As commented in Section 4, when applied to autonomous systems, our findings refine the approach in (Kiendl *et al*, 1992) by pointing out the decreasing rates of the contractive invariant sets, in the sense that (Kiendl *et al*, 1992) deals only with constant invariant sets. Our results also encompass, as a particular case, the analysis of exponentially decreasing invariant sets defined by diagonal matrices, which was developed in (Gruyitch *et al*, 2004), (Pastravanu and Voicu, 2006) with respect to autonomous linear systems.

2. THE CONTINUOUS TIME CASE

Let us consider the time-varying linear system described in CT by the differential equation:

$$\begin{gathered} \dot{x}(t) = A(t)\,x(t),\; x(t_0) = x_0 \in \mathbf{R}^n , \\ t, t_0 \in \mathbf{R}_+ ,\; t \ge t_0 , \end{gathered} \tag{8}$$

where $A(t) \in \mathbf{R}^{n \times n}$ is a continuous matrix function. In the state space of non-autonomous system (8), let us consider the time-dependent sets defined by:

$$\begin{gathered} X_{p,G,r}^c(t) = \left\{ x \in \mathbf{R}^n \,\middle|\, \| Gx \|_p \le c e^{rt} \right\}, \\ c > 0,\; t \in \mathbf{R}_+ , \end{gathered} \tag{9}$$

where $G \in \boldsymbol{R}^{n \times n}$ is a nonsingular matrix and $r < 0$.

Definition 1. A set $X_{p,G,r}^c(t)$ defined by (9) is *flow* (*positively*) *invariant with respect to* system (8), if any trajectory initiated inside the set remains inside it at any time, i.e

$$\begin{gathered} \forall t_0 \in \mathbf{R}_+ , \forall x_0 \in X_{p,G,r}^c(t_0) \Rightarrow \\ \Rightarrow \forall t \in \mathbf{R}_+ ,\, t > t_0 ,\, x(t; t_0, x_0) \in X_{p,G,r}^c(t). \end{gathered} \quad \blacksquare \tag{10}$$

In accordance with the nomenclature in (Blanchini, 1999) and (Gruyitch *et al*, 2004), a set of form (9) is said to be *contractive*.

Theorem 1. Let $1 \le p \le \infty$. A contractive set $X_{p,G,r}^c(t)$ given by (9) is positively invariant with respect to system (8) if and only if

$$\forall t \in \mathbf{R}_+ ,\; m_p(GA(t)G^{-1}) \le r < 0 . \tag{11}$$

Proof. The positive invariance of the sets (9) with respect to system (8) is equivalent with the invariance of the set

$$Y_p = \left\{ y \in \mathbf{R}^n \,\middle|\, \| y \|_p \le 1 \right\} , \tag{12}$$

with respect to the system

$$\dot{y}(t) = C(t)y(t),\ y(t_0) = y_0 \in \mathbf{R}^n,\ t, t_0 \in \mathbf{R}_+,\ t \ge t_0, \quad (13)$$

where

$$C(t) = -rI + GA(t)G^{-1},\ t \in \mathbf{R}_+. \quad (14)$$

This equivalence is motivated by the fact that system (8) and system (13) are mutually related by the time-dependent nonsingular transformation

$$y(t) = (ce^{rt})^{-1} G\, x(t),\ t \in \mathbf{R}_+. \quad (15)$$

Hence, the proof of Theorem 1 can be reduced to the case of system (13), and we have to show that the inequality

$$\forall t \in \mathbf{R}_+,\ m_p(C(t)) \le 0, \quad (16)$$

is a necessary and sufficient condition for the positive invariance of the set (12) with respect to system (14). Obviously, inequality (16) is equivalent to inequality (11).

Necessity. The invariance of the set (12) with respect to system (13) implies that the function

$$V_p(y): \mathbf{R}^n \to \mathbf{R}_+,\ V_p(y) = \| y \|_p, \quad (17)$$

is nonincreasing along each trajectory of system (13). Indeed, let y be a solution to (13) and let $t, t_0 \in \mathbf{R}_+$, $t \ge t_0$. Set $\varepsilon = \| y(t_0) \|_p$. If $\varepsilon = 0$, then $y(t_0) = 0$, and hence $y(t) = 0$, meaning that $V_p(y(t)) = V_p(y(t_0)) = 0$ for $t \ge t_0$. If $\varepsilon > 0$, then $\bar{y} = \varepsilon^{-1} y$ is also a solution to (13) and $\| \bar{y}(t_0) \|_p = 1$. Since the set Y_p defined by (12) is invariant with respect to system (13), we have $\| \bar{y}(t) \|_p \le 1$, or, equivalently $\| \varepsilon^{-1} y(t) \|_p \le 1$, such that, we finally obtain the non-increasing monotonicity of the function $V_p(t)$, i.e. $V_p(y(t)) = \| y(t) \|_p \le \varepsilon = \| y(t_0) \|_p = V_p(y(t_0))$ for all $t, t_0 \in \mathbf{R}_+$, $t \ge t_0$.

As an intermediary step, let us show that the transition matrix of system (13), denoted by $\mathbf{\Psi}(t, t_0)$, fulfills the inequality:

$$\forall t, t_0 \in \mathbf{R}_+,\ t \ge t_0,\ \| \mathbf{\Psi}(t, t_0) \|_p \le 1. \quad (18)$$

Consider arbitrary $t, t_0 \in \mathbf{R}_+$, $t \ge t_0$. There exists a vector $y_0 \in \mathbf{R}^n$, $\| y_0 \|_p = 1$, such that $\| \mathbf{\Psi}(t, t_0) \|_p = \| \mathbf{\Psi}(t, t_0) y_0 \|_p$. If $y(t_0) = y_0$, then for $y(t) = \mathbf{\Psi}(t, t_0) y(t_0)$ we have $\|y(t)\|_p \le \|y(t_0)\|_p = 1$, due to the nonincreasing monotonicity of $V_p(y) = \| y \|_p$ along any trajectory of system (13). Consequently, $\| \mathbf{\Psi}(t, t_0) \|_p = \| y(t) \|_p \le 1$.

Now we shall prove inequality (16), but towards this end, we need to show that

$$\forall t \in \mathbf{R}_+,\ \lim_{\theta \downarrow 0} \frac{\| \mathbf{\Psi}(t+\theta, t) \|_p - 1}{\theta} = m_p(C(t)). \quad (19)$$

Indeed, if $M(t)$ denotes a fundamental matrix of system (13), satisfying $\dot{M}(t) = C(t)M(t)$, then $\mathbf{\Psi}(t+\theta, t) = M(t+\theta)M^{-1}(t) = [M(t) + \theta \dot{M}(t) + \theta O(\theta)]M^{-1}(t) = I + \theta C(t) + \theta O(\theta) M^{-1}(t)$, for all $\theta \ge 0$, where $\lim_{\theta \downarrow 0} O(\theta) = 0$. On the other hand, $\| I + \theta C(t) \|_p - \theta \| O(\theta) M^{-1}(t) \|_p \le \| I + \theta C(t) + \theta O(\theta) M^{-1}(t) \|_p \le \| I + \theta C(t) \|_p + \theta \| O(\theta) M^{-1}(t) \|_p$, and we can write $\frac{\|I + \theta C(t)\|_p - 1}{\theta} - \|O(\theta) M^{-1}(t)\|_p \le \frac{\|\mathbf{\Psi}(t+\theta, t)\|_p - 1}{\theta} \le \frac{\|I + \theta C(t)\|_p - 1}{\theta} + \|O(\theta) M^{-1}(t)\|_p$, which yields (19). Since $\| \mathbf{\Psi}(t+\theta, t) \|_p \le 1$ for any $t, \theta \in \mathbf{R}_+$, from (19) we get $m_p(C(t)) \le 0$.

Sufficiency. Let y be a solution to (13) and let $t \in \mathbf{R}_+$. Thus,

$$\lim_{\theta \downarrow 0} \frac{V_p(y(t+\theta)) - V_p(y(t))}{\theta} = \lim_{\theta \downarrow 0} \frac{\| \mathbf{\Psi}(t+\theta, t) y(t) \|_p - \| y(t) \|_p}{\theta} \le \lim_{\theta \downarrow 0} \frac{\| \mathbf{\Psi}(t+\theta, t) \|_p \| y(t) \|_p - \| y(t) \|_p}{\theta} = \lim_{\theta \downarrow 0} \frac{\|\mathbf{\Psi}(t+\theta, t)\|_p - 1}{\theta} \| y(t) \|_p = m_p(C(t)) \| y(t) \|_p \le 0$$

meaning that $V_p(y(t))$ is nonincreasing along each trajectory of (13) if condition (16) is satisfied.

Assume, by contradiction, that the set Y_p defined by (12) is not invariant with respect to system (13). Let y solve (13) and violate the invariance condition, i.e. there exists $t^*, t^{**} \in \mathbf{R}_+$, $t^* < t^{**}$ such that $\|y(t^*)\|_p \le 1$ and $\|y(t^{**})\|_p > 1$. This means $V_p(y(t^{**})) > V_p(y(t^*))$ which contradicts the hypothesis that $V_p(y(t))$ is nonincreasing along each trajectory of system (13). ∎

On the same mathematical bases, a separate result can be derived for constant invariant sets of form (4).

Corollary 1. Let $1 \le p \le \infty$. A constant set $X_{p,G}^{c}$ of form (4) is positively invariant with respect to system (8) if and only if

$$m_p(GA(t)G^{-1}) \le 0 . \tag{20}$$

Proof. It is constructed along the same lines as the proof of Theorem 1, by considering $r = 0$. ■

3. THE DISCRETE TIME CASE

Let us consider the DT non-autonomous linear system described by the difference equation:

$$x(t+1) = A(t)x(t),\ x(t_0) = x_0 \in \mathbf{R}^n ,\quad t, t_0 \in \mathbf{Z}_+ ,\ t \ge t_0 , \tag{21}$$

where $A(t) \in \mathbf{R}^{n\times n}$. In the state space of the time-varying system (21), let us consider the time-dependent sets defined by:

$$X_{p,G,r}^{c}(t) = \left\{ x \in \mathbf{R}^n \,\middle|\, \| Gx \|_p \le cr^t \right\},\quad c > 0,\ t \in \mathbf{Z}_+, \tag{22}$$

where $G \in \boldsymbol{R}^{n\times n}$ is a nonsingular matrix and $0 < r < 1$. As stated in the introductory part of our paper, the approach to the CT case goes *mutatis mutandis* in the DT case. For that reason, we will only outline the necessary adjustments of the framework presented in Section 2.

Definition 2. A set $X_{p,G,r}^{c}(t)$ defined by (22) is *flow* (*positively*) *invariant with respect to* system (21), if any trajectory initiated inside the set remains inside it at any time, i.e

$$\forall t_0 \in \mathbf{Z}_+, \forall x_0 \in X_{p,G,r}^{c}(t_0) \Rightarrow \Rightarrow \forall t \in \mathbf{Z}_+,\ t > t_0,\ x(t;t_0,x_0) \in X_{p,G,r}^{c}(t). \quad ■ \tag{23}$$

Similar to the CT case, a set of form (22) is said to be *contractive*.

Theorem 2. Let $1 \le p \le \infty$. A contractive set $X_{p,G,r}^{c}(t)$ defined by (22) is positively invariant with respect to system (21) if and only if

$$\forall t \in \mathbf{Z}_+,\ \| GA(t)G^{-1} \|_p \le r < 1 . \tag{24}$$

Proof. The positive invariance of the sets (22) with respect to system (21) is equivalent with the invariance of the set $Y_p = \left\{ y \in \mathbf{R}^n \,\middle|\, \| y \|_p \le 1 \right\}$ defined by (12) with respect to the system

$$y(t+1) = C(t)y(t),\ y(t_0) = y_0 \in \mathbf{R}^n ,\quad t, t_0 \in \mathbf{Z}_+ ,\ t \ge t_0 , \tag{25}$$

where

$$C(t) = r^{-1}GA(t)G^{-1},\ t \in \mathbf{Z}_+ . \tag{26}$$

Note that systems (21) and (25) are mutually related by the time-dependent nonsingular transformation

$$y(t) = (cr^t)^{-1} G\,x(t),\ t \in \mathbf{Z}_+ . \tag{27}$$

The proof of Theorem 2 can be reduced to the case of system (25), and we have to show that inequality

$$\forall t \in \mathbf{Z}_+,\ \| C(t) \|_p \le 1 , \tag{28}$$

is a necessary and sufficient condition for the positive invariance of the set (12) with respect to system (25). Obviously, inequalities (28) and (24) are equivalent.

Necessity. The invariance of the set (12) with respect to system (25) implies that the function $V_p(y): \mathbf{R}^n \to \mathbf{R}_+$, $V_p(y) = \| y \|_p$, defined by (17), is nonincreasing along each trajectory of system (25). Consider arbitrary $t = t_0 \in \mathbf{Z}_+$. There exists a vector $y_0 \in \mathbf{R}^n$, $\| y_0 \|_p = 1$, such that $\|C(t)\|_p = \|C(t)y_0\|_p$. If $y(t) = y_0$, then $y(t+1) = C(t)y_0$. Due to the nonincreasing monotonicity of $V_p(y) = \| y \|_p$ along any trajectory of system (25), we have $\|y(t+1)\|_p \le \|y(t)\|_p = \| y_0 \|_p = 1$. Thus, $\|C(t)\|_p = \|y(t+1)\|_p \le 1$.

Sufficiency. Let y be a solution to (25) and let $t \in \mathbf{Z}_+$. Hence, $V_p(y(t+1)) = \|C(t)y(t)\|_p \le \|C(t)\|_p \|y(t)\|_p \le \| y(t) \|_p = V_p(y(t))$, meaning that function $V_p(y)$ is nonincreasing along each trajectory of (25). Assume, by contradiction, that the set Y_p defined by (12) is not invariant with respect to system (25). Therefore a solution y to (25) can be found that violates the invariance condition, i.e. there exist $t^*, t^{**} \in \mathbf{R}_+$, $t^* < t^{**}$ such that $\|y(t^*)\|_p \le 1$ and $\|y(t^{**})\|_p > 1$. This implies $V_p(y(t^{**})) > V_p(y(t^*))$, contradicting the hypothesis that $V_p(y(t))$ is nonincreasing along each trajectory of system (25). ■

For constant invariant sets of form (4), the following result can be derived.

Corollary 2. Let $1 \le p \le \infty$. A constant set $X^c_{p,G}$ of form (4) is positively invariant with respect to DT system (22) if and only if

$$\| GA(t)G^{-1} \|_p \le 1 . \qquad (29)$$

Proof. It is constructed along the same lines as the proof of Theorem 2, by considering $r = 1$. ■

4. LINKS WITH PREVIOUS RESULTS

Remark 1. As expected, Corollary 1 in the CT case, accommodates the invariance problems addressed by (Kiendl, 1992) for the autonomous system (1) as a particular situation. Moreover, unlike (Kiendl, 1992) providing a sufficient condition by the strict inequality (2), Corollary 1 shows that the non-strict inequality (20) is necessary and sufficient for constant invariant sets. As a matter of fact, Theorem 1 particularized for the autonomous systems (1) suggests that the strict inequality (2) used in (Kiendl, 1992) represents a necessary and sufficient condition for contractive invariant sets. Similarly, in the DT case, Corollary 2 accommodates the invariance problems addressed by (Kiendl, 1992) for the autonomous system (5) as a particular situation. Moreover, Corollary 2 shows that the non-strict inequality (29) is necessary and sufficient for constant invariant sets, unlike (Kiendl, 1992) that provides a sufficient condition by the strict inequality (6). Theorem 2 particularized for the autonomous systems (5) suggests that the strict inequality (6) used in (Kiendl, 1992) represents a necessary and sufficient condition for contractive invariant sets. ■

Remark 2. The incomplete interpretation of the strict inequalities (2) and (6) given in (Kiendl, 1992) is due to the main objective of (Kiendl, 1992) looking for strong Lyapunov functions for the autonomous systems (1) and (5). Based on the proofs presented above for Theorems 1 and 2 (whose principles are also valid for the proofs of the two Corollaries) we are able to offer further details. For the sake of brevity, these particulars are handed only in the case of CT systems, but they can easily be paralleled to the case of DT systems.

(i) In the case of constant invariant sets of form (4), the equivalence transformation (15) between systems (8) and (13) with $r = 0$ yields $\| Gx(t) \|_p = cV_p(y(t))$, which, together with the non-increasing monotonicity of $V_p(y)$ defined by (17), shows that

$$W_p(x) : \mathbf{R}^n \to \mathbf{R}_+,\, W_p(x) = \| Gx \|_p \qquad (30)$$

is nonincreasing along each trajectory of the non-autonomous system (8). Since G is a full rank matrix, $W_p(x)$ is positive definite, and, consequently, it is a weak Lyapunov function. Hence, the equilibrium $x = \{0\}$ of system (8) is stable. This is in full accordance with the non-strict inequality (21) which provides all nonsingular matrices $G \in \boldsymbol{R}^{n\times n}$ defining constant sets of form (4), invariant with respect to the linear system (8).

(ii) In the case of contractive invariant sets of form (9), the time-dependent equivalence transformation (15) between systems (8) and (13) with $r < 0$ yields $\| Gx(t) \|_p = ce^{rt}V_p(y(t))$, which, together with the non-increasing monotonicity of $V_p(y)$ defined by (17), shows that $W_p(x)$ defined by (30) is decreasing along each trajectory of the non-autonomous system (8). Since G is a full rank matrix, $W_p(x)$ is positive definite, and, consequently, it is a strong Lyapunov function. Hence, the equilibrium $x = \{0\}$ of system (8) is asymptotically stable. This is in full accordance with the strict inequality (16), or, equivalently (11), which provides all nonsingular matrices $G \in \mathbf{R}^{n\times n}$ defining contractive sets of form (9), invariant with respect to the linear system (8). The discussed inequality in form (11) includes the concrete value of the decreasing factor $r < 0$ that describes the contraction of the invariant set; this explains the differences between the invariant sets built with various matrices G that satisfy inequality (11). ■

Remark 3. Theorems 1 also encompasses, as particular cases, the results formulated for CT autonomous systems in (Gruyitch *et al*, 2004) which considered contractive invariant sets of form (9) defined by diagonal matrices G with positive entries, for $p \in \{1, 2, \infty\}$. Along the same line of remark, the results presented for CT and DT autonomous systems in (Pastravanu and Voicu, 2006), which considered contractive invariant sets of form (9) and (22), respectively, defined by diagonal matrices G with positive entries, for $1 \le p \le \infty$, are also covered by Theorems 1 and 2 respectively. ■

5. CONCLUDING REMARKS

The theorems and corollaries presented in this paper provide criteria for testing or constructing sets which are invariant with respect to the trajectories of non-autonomous linear systems in CT or DT. These criteria accommodate, as particular cases, the results already known for autonomous linear systems. From the methodological point of view, our approach points out the role of matrix measures in the analysis of the invariant sets for CT systems, meaning a reformulation of the classical tangency condition in terms of the operators defining the linear systems dynamics. For revealing the invariance properties of DT systems, a parallel development can be devised based on matrix norms.

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