THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
FASCICLE III, 2004 ISSN 1221-454X
ELECTROTEHNICS, ELECTRONICS, AUTOMATIC CONTROL, INFORMATICS

COMPONENTWISE STABILITY OF DISCRETE-TIME INTERVAL BIDIRECTIONAL ASSOCIATIVE MEMORIES

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Abstract: The componentwise stability is a special type of asymptotic stability, which incorporates the positive invariance of certain time-dependent rectangular sets with respect to the state space trajectories. The paper develops the analysis of componentwise stability for discrete-time Bidirectional Associative Memory (BAM) neural networks with interval type parameters, providing criteria that allow monitoring the evolution of each state-space variable towards the equilibrium point. These criteria are formulated in terms of Schur stability of a test matrix adequately built from the intervals expressing the parameter uncertainties. Our approach represents a refinement of the classical results in stability theory, since the time-dependence of the considered invariant sets makes it possible to give a qualitative characterization of the dynamics at the level of the state vector components.

Keywords: stability analysis, neural networks, discrete-time systems, nonlinear systems.

1. INTRODUCTION

Consider the discrete-time *bidirectional associative memory* (BAM) neural network described by

$$(1)\quad \begin{aligned} &\boldsymbol{x}^1(t+1)=\boldsymbol{A}^1\boldsymbol{x}^1(t)+\boldsymbol{W}^2\boldsymbol{f}^2\left(\boldsymbol{x}^2(t)\right)+\boldsymbol{I}^1, \\ &\boldsymbol{x}^2(t+1)=\boldsymbol{A}^2\boldsymbol{x}^2(t)+\boldsymbol{W}^1\boldsymbol{f}^1\left(\boldsymbol{x}^1(t)\right)+\boldsymbol{I}^2, \\ &\boldsymbol{x}^1(t_0)=\boldsymbol{x}^1_0,\quad \boldsymbol{x}^2(t_0)=\boldsymbol{x}^2_0,\quad t_0\in\mathbb{Z}_+, \end{aligned}\quad t\in\mathbb{Z}_+,$$

where $\boldsymbol{x}^1=\left[x^1_1,x^1_2,\ldots,x^1_m\right]^\tau$, $\boldsymbol{x}^2=\left[x^2_1,x^2_2,\ldots,x^2_n\right]^\tau$ are the state-vectors (τ denoting the vector transposition), $\boldsymbol{I}^1=\left[I^1_1,I^1_2,\ldots,I^1_m\right]^\tau$, $\boldsymbol{I}^2=\left[I^2_1,I^2_2,\ldots,I^2_n\right]^\tau$ are the input vectors and matrices $\boldsymbol{A}^1=\mathrm{diag}\left\{a^1_1,a^1_2,\ldots,a^1_m\right\}$, $\boldsymbol{A}^2=\mathrm{diag}\left\{a^2_1,a^2_1,\ldots,a^2_n\right\}$, $\boldsymbol{W}^1=\left[w^1_{ji}\right]$, $\boldsymbol{W}^2=\left[w^2_{ij}\right]$ have appropriate sizes.

All the components of the activation functions $\boldsymbol{f}^1:\mathbb{R}^m\to\mathbb{R}^m$, $\boldsymbol{f}^1(\boldsymbol{x}^1)=\left[f^1_1(x^1_1),f^1_2(x^1_2),\ldots,f^1_m(x^1_m)\right]^\tau$, $\boldsymbol{f}^2:\mathbb{R}^n\to\mathbb{R}^n$, $\boldsymbol{f}^2(\boldsymbol{x}^2)=\left[f^2_1(x^2_1),f^2_2(x^2_2),\ldots,f^2_n(x^2_n)\right]^\tau$ are globally Lipschitz continuous, i.e. for all $i=\overline{1,m}$, $j=\overline{1,n}$, there exist $L^1_i,L^2_j>0$ so that

$$(2)\quad \begin{aligned} &0\le\left|f^1_i(\xi)-f^1_i(\zeta)\right|\le L^1_i\,|\,\xi-\zeta\,|, \\ &0\le\left|f^2_j(\xi)-f^2_j(\zeta)\right|\le L^2_j\,|\,\xi-\zeta\,|, \end{aligned},$$

for all $\xi,\zeta\in\mathbb{R}$.

In terms of neural networks, according to our hypotheses, the activation functions are assumed to be neither differentiable nor bounded. If f is the generic notation for an activation function, our approach covers the following classes of functions: bipolar

This paper was recommended for publication by Viorel MINZU

sigmoid $f(s)=[1-\exp(-\lambda s)]/[1+\exp(-\lambda s)]$, $\lambda>0$, piecewise saturation $f(s)=[|\lambda s+1|-|\lambda s-1|]/2$, $\lambda>0$, linear $f(s)=\lambda s, \lambda>0$, piecewise linear, etc.

Recent papers, such as (Cao *et al*., 2002), (Liao *et al*., 2002), (Mohamad, 2001) and (Zhang *et al.*, 2001), provide sufficient conditions, formulated in algebraic terms, for the global (exponential) asymptotic stability of continuous-time BAMs. Work (Mohamad, 2001) also refers to the discrete-time case.

Taking into consideration the nature of the mathematical problems discussed in this paper, the following notations are used in the remainder of our paper. $\boldsymbol{O}$ and $\boldsymbol{0}$ stand for the null square matrix and the null vector respectively, each of appropriate dimensions. For two matrices having the same sizes $\Phi=[\phi_{rs}]$, $\Theta=[\theta_{rs}]\in\mathbb{R}^{u\times v}$, matrix inequality $\Phi\geq\Theta$ ($\Phi>\Theta$) is understood componentwise, i.e. $\phi_{rs}\geq\theta_{rs}$ ($\phi_{rs}>\theta_{rs}$) for all $r=\overline{1,u}$ and $s=\overline{1,v}$. Given a matrix $\Phi=[\phi_{rs}]\in\mathbb{R}^{u\times v}$, denote by $|\Phi|$ the matrix whose entries are $|\phi_{rs}|$. These conventions apply in the case of vectors or vector functions too.

For many problems encountered in practice it is important to consider that the entries of the matrices $\boldsymbol{A}^k$, $\boldsymbol{W}^k$, $k=\overline{1,2}$, defining the dynamics of BAM (1), are uncertain, in the sense of the matrix componentwise inequalities:

$$\text{(3)}\quad \begin{aligned} &\underline{\boldsymbol{A}}^1=\operatorname{diag}\{\underline{a}^1_1,\underline{a}^1_2,\dots,\underline{a}^1_m\}\leq\boldsymbol{A}^1\leq\\ &\qquad\leq\overline{\boldsymbol{A}}^1=\operatorname{diag}\{\overline{a}^1_1,\overline{a}^1_2,\dots,\overline{a}^1_m\},\\ &\underline{\boldsymbol{A}}^2=\operatorname{diag}\{\underline{a}^2_1,\underline{a}^2_2,\dots,\underline{a}^2_n\}\leq\boldsymbol{A}^2\leq\\ &\qquad\leq\overline{\boldsymbol{A}}^2=\operatorname{diag}\{\overline{a}^2_1,\overline{a}^2_2,\dots,\overline{a}^2_n\},\\ &\underline{\boldsymbol{W}}^1=\left[\underline{w}^1_{ji}\right]_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}}\leq\boldsymbol{W}^1\leq\overline{\boldsymbol{W}}^1=\left[\overline{w}^1_{ji}\right]_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}},\\ &\underline{\boldsymbol{W}}^2=\left[\underline{w}^2_{ij}\right]_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}}\leq\boldsymbol{W}^2\leq\overline{\boldsymbol{W}}^2=\left[\overline{w}^2_{ij}\right]_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}}.\end{aligned}$$

Consequently, let us introduce the following classes of matrices:

$$\text{(4)}\quad \begin{aligned} &\mathcal{A}^1=\left\{\boldsymbol{A}^1\in\mathbb{R}^{m\times m}\,\middle|\,\underline{\boldsymbol{A}}^1\leq\boldsymbol{A}^1\leq\overline{\boldsymbol{A}}^1\right\},\\ &\mathcal{A}^2=\left\{\boldsymbol{A}^2\in\mathbb{R}^{n\times n}\,\middle|\,\underline{\boldsymbol{A}}^2\leq\boldsymbol{A}^2\leq\overline{\boldsymbol{A}}^2\right\},\\ &\mathcal{W}^1=\left\{\boldsymbol{W}^1\in\mathbb{R}^{n\times m}\,\middle|\,\underline{\boldsymbol{W}}^1\leq\boldsymbol{W}^1\leq\overline{\boldsymbol{W}}^1\right\},\\ &\mathcal{W}^2=\left\{\boldsymbol{W}^2\in\mathbb{R}^{m\times n}\,\middle|\,\underline{\boldsymbol{W}}^2\leq\boldsymbol{W}^2\leq\overline{\boldsymbol{W}}^2\right\}.\end{aligned}$$

The family of BAMs generated by (1) for all $\boldsymbol{A}^k\in\mathcal{A}^k$, $\boldsymbol{W}^k\in\mathcal{W}^k$, $k=\overline{1,2}$, is called *Interval Bidirectional Associative Memory* neural network, abbreviated as IBAM($\mathcal{A}^1$,$\mathcal{A}^2$,$\mathcal{W}^1$,$\mathcal{W}^2$).

This paper proves that the stability of a single test matrix guarantees a stronger stability property of IBAM($\mathcal{A}^1$,$\mathcal{A}^2$,$\mathcal{W}^1$,$\mathcal{W}^2$), called *componentwise stability*. Unlike the standard concepts of stability, that give global information on the state-space vector, expressed in terms of arbitrary norms, the componentwise stability allows an individual monitoring of each state-space variable. This type of stability was first studied by Voicu in (Voicu, 1984) who applied the theory of flow-invariant time-dependent rectangular sets to define and characterize the *componentwise asymptotic stability* (CWAS) and the *componentwise exponential asymptotic stability* (CWEAS) for continuous-time linear systems. Further works extended the analysis of componentwise stability to continuous-time delay linear systems (Hmamed, 1996), 1-D and 2-D discrete-time linear systems (Hmamed, 1997), interval matrix systems (Pastravanu and Voicu, 2002) and a class of Persidskii systems with uncertainties (Pastravanu and Voicu, 2003). Despite the existence of these results, the componentwise stability of recurrent neural networks remained almost unexplored, except for a reduced number of recent papers (Chu *et al.*, 2002a), (Chu *et al.*, 2002b), (Chu *et al.*, 2003) and (Matcovschi and Pastravanu, 2003).

Our paper develops a CWAS/CWEAS analysis of BAM (1) under the hypothesis of parameter uncertainties modeled by $\boldsymbol{A}^k\in\mathcal{A}^k$, $\boldsymbol{W}^k\in\mathcal{W}^k$, $k=\overline{1,2}$. The concepts employed by our work are rigorously defined in Section II. Section III provides the main results, consisting in sufficient criteria for the CWAS/CWEAS of BAMs with uncertainties. Section IV creates a deeper insight into some frequently encountered particular cases and allows comparisons with other papers. A few final remarks are formulated in Section V. All over the text, the vector (matrix) inequalities have componentwise meaning.

2. PRELIMINARIES

Assume that BAM (1) has a finite number of equilibrium points and let $\boldsymbol{x}_e=\left[\boldsymbol{x}_e^{1\tau}\;\boldsymbol{x}_e^{2\tau}\right]^\tau$ be one of these, i.e. $\boldsymbol{x}^1_e=\boldsymbol{A}^1\boldsymbol{x}^1_e+\boldsymbol{W}^2\boldsymbol{f}^2\left(\boldsymbol{x}^2_e\right)+\boldsymbol{I}^1$ and $\boldsymbol{x}^2_e=\boldsymbol{A}^2\boldsymbol{x}^2_e+\boldsymbol{W}^1\boldsymbol{f}^1\left(\boldsymbol{x}^1_e\right)+\boldsymbol{I}^2$.

Definition 1. (**a**) Let $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ be two vector functions $\boldsymbol{p}^1:\mathbb{R}_+\to\mathbb{R}^m$, $\boldsymbol{p}^2:\mathbb{R}_+\to\mathbb{R}^n$, with positive components $p^1_i(t)>0$, $i=\overline{1,m}$, $p^2_j(t)>0$, $j=\overline{1,n}$, $t\in\mathbb{R}_+$, meeting

$$\text{(5)}\quad \lim_{t\to\infty}\boldsymbol{p}^k(t)=\boldsymbol{0},\ \ k=\overline{1,2}.$$

If for any $t_0 \in \mathbb{R}_+$ and any initial condition $\boldsymbol{x}_0 = \left[\boldsymbol{x}_0^{1\tau}\ \boldsymbol{x}_0^{2\tau}\right]^\tau$, $\boldsymbol{x}_0^1 \in \mathbb{R}^m$, $\boldsymbol{x}_0^2 \in \mathbb{R}^n$, satisfying $\left|\boldsymbol{x}_0^k - \boldsymbol{x}_e^k\right| \le \boldsymbol{p}^k(t_0)$, $k = \overline{1,2}$, the corresponding solution to (1), $\boldsymbol{x}(t) = \left[\boldsymbol{x}^1(t)^\tau\ \boldsymbol{x}^2(t)^\tau\right]^\tau$, $\boldsymbol{x}^k(t) = \boldsymbol{x}^k(t;t_0,\boldsymbol{x}_0)$, $k = \overline{1,2}$, meets the inequality $\left|\boldsymbol{x}^k(t) - \boldsymbol{x}_e^k\right| \le \boldsymbol{p}^k(t)$, $\forall t \in \mathbb{R}_+$, $t \ge t_0$, $k = \overline{1,2}$, then we say that the equilibrium point $\boldsymbol{x}_e$ of BAM (1) is *componentwise asymptotically stable* with respect to $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$, abbreviated as CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$).

(**b**) The equilibrium point $\boldsymbol{x}_e$ of BAM (1) is *globally* CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), or CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) *in the large*, abbreviated as GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), if $\boldsymbol{x}_e$ is CWAS($c\boldsymbol{p}^1, c\boldsymbol{p}^2$) for any scalar $c > 0$.

(**c**) BAM (1) is said to be CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) if it has an equilibrium point $\boldsymbol{x}_e$ that is GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$).

(**d**) IBAM($\mathcal{A}^1, \mathcal{A}^2, \mathcal{W}^1, \mathcal{W}^2$) is said to be CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) if BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $k = \overline{1,2}$.

Remark 1. It can be proved that (**a**) if an equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), then it is also *uniformly asymptotically stable* in the sense of the standard definition, e.g. (Michel and Wang, 1995), pp. 107; (**b**) if an equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), then it is also *uniformly asymptotically stable in the large* in the sense of the standard definition, e.g. (Michel and Wang, 1995), pp. 108).

Until this point of our presentation the time-dependence of the vector functions $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, was considered arbitrary. If the CWAS property exists for the particular form of the vector functions

$$\text{(6)} \quad \begin{aligned} &\boldsymbol{p}^1(t) = \sigma^t \boldsymbol{\alpha}^1,\ \boldsymbol{p}^2(t) = \sigma^t \boldsymbol{\alpha}^2,\ t \in \mathbb{R}_+, \\ &\sigma \in (0,1),\ \boldsymbol{\alpha}^1 \in \mathbb{R}^m,\ \boldsymbol{\alpha}^1 > \boldsymbol{0},\ \boldsymbol{\alpha}^2 \in \mathbb{R}^n,\ \boldsymbol{\alpha}^2 > \boldsymbol{0}, \end{aligned}$$

then we refer to a special type of stability property called *componentwise exponential asymptotic stability*, abbreviated as CWEAS, and Definition 1 yields the following:

Definition 2. If the hypotheses of Definition 1(a-d) are fulfilled with $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, given by (6), then we say that: (**a**) the equilibrium point $\boldsymbol{x}_e$ is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$); (**b**) the equilibrium point $\boldsymbol{x}_e$ is globally CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), abbreviated as GCWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$); (**c**) BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$);(**d**) IBAM($\mathcal{A}^1, \mathcal{A}^2, \mathcal{W}^1, \mathcal{W}^2$) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$).

Remark 2. It can be proved that (**a**) if an equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), then it is also *exponentially asymptotically stable* in the classical sense (e.g. (Michel and Wang, 1995), pp. 107); (**b**) if an equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is GCWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), then it is also *globally exponentially asymptotically stable* in the classical sense (e.g. (Michel and Wang, 1995), pp. 108).

3. MAIN RESULTS

3.1. CWAS of IBAMs

Theorem 1. IBAM($\mathcal{A}^1, \mathcal{A}^2, \mathcal{W}^1, \mathcal{W}^2$) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) if the following inequalities hold

$$\text{(7)} \quad \begin{aligned} &\boldsymbol{p}^1(t+1) \ge \hat{\boldsymbol{A}}^1 \boldsymbol{p}^1(t) + \hat{\boldsymbol{W}}^2 \boldsymbol{L}^2 \boldsymbol{p}^2(t), \\ &\boldsymbol{p}^2(t+1) \ge \hat{\boldsymbol{A}}^2 \boldsymbol{p}^2(t) + \hat{\boldsymbol{W}}^1 \boldsymbol{L}^1 \boldsymbol{p}^1(t), \end{aligned} \quad \forall t \in \mathbb{R}_+,$$

where matrices $\hat{\boldsymbol{A}}^k$, $\hat{\boldsymbol{W}}^k$, $\boldsymbol{L}^k$, $k = \overline{1,2}$, are defined by

$$\hat{\boldsymbol{A}}^1 = \mathrm{diag}\left\{\hat{a}_1^1, \hat{a}_2^1, \ldots, \hat{a}_m^1\right\},\ \hat{a}_i^1 = \max\left\{|\underline{a}_i^1|, |\overline{a}_i^1|\right\},\ i = \overline{1,m},$$
$$\hat{\boldsymbol{A}}^2 = \mathrm{diag}\left\{\hat{a}_1^2, \hat{a}_2^2, \ldots, \hat{a}_n^2\right\},\ \hat{a}_j^2 = \max\left\{|\underline{a}_j^2|, |\overline{a}_j^2|\right\},\ j = \overline{1,n},$$

$$\text{(8)} \quad \begin{aligned} &\hat{\boldsymbol{W}}^1 = \left[\hat{w}_{ji}^1\right] \in \mathbb{R}^{n\times m},\ \hat{w}_{ji}^1 = \max\left\{|\underline{w}_{ji}^1|, |\overline{w}_{ji}^1|\right\}, \\ &\hat{\boldsymbol{W}}^2 = \left[\hat{w}_{ij}^2\right] \in \mathbb{R}^{m\times n},\ \hat{w}_{ij}^2 = \max\left\{|\underline{w}_{ij}^2|, |\overline{w}_{ij}^2|\right\}, \\ &\boldsymbol{L}^1 = \mathrm{diag}\left\{L_1^1, L_2^1, \ldots, L_m^1\right\} \in \mathbb{R}^{m\times m}, \\ &\boldsymbol{L}^2 = \mathrm{diag}\left\{L_1^2, L_2^2, \ldots, L_n^2\right\} \in \mathbb{R}^{n\times n}. \end{aligned}$$

Proof: Given arbitrary $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $k = \overline{1,2}$, the dynamical behavior of the state-space trajectories of BAM (1) in a vicinity of the equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ may be analyzed by means of the deviations $\boldsymbol{y}^k(t) = \boldsymbol{x}^k(t) - \boldsymbol{x}_e^k$, $k = \overline{1,2}$, that satisfy

$$\text{(9)} \quad \begin{aligned} &\boldsymbol{y}^1(t+1) = \boldsymbol{A}^1 \boldsymbol{y}^1(t) + \boldsymbol{W}^2 \boldsymbol{g}^2\left(\boldsymbol{y}^2(t)\right), \\ &\boldsymbol{y}^2(t+1) = \boldsymbol{A}^2 \boldsymbol{y}^2(t) + \boldsymbol{W}^1 \boldsymbol{g}^1\left(\boldsymbol{y}^1(t)\right), \\ &\boldsymbol{y}^1(t_0) = \boldsymbol{x}_0^1 - \boldsymbol{x}_e^1,\quad \boldsymbol{y}^2(t_0) = \boldsymbol{x}_0^2 - \boldsymbol{x}_e^2, \end{aligned}$$

where

$$\text{(10)} \quad \begin{aligned} &\boldsymbol{g}^1(\boldsymbol{y}^1) = \boldsymbol{f}^1(\boldsymbol{y}^1 + \boldsymbol{x}_e^1) - \boldsymbol{f}^1(\boldsymbol{x}_e^1), \\ &\boldsymbol{g}^2(\boldsymbol{y}^2) = \boldsymbol{f}^2(\boldsymbol{y}^2 + \boldsymbol{x}_e^2) - \boldsymbol{f}^2(\boldsymbol{x}_e^2). \end{aligned}$$

Obviously, $\boldsymbol{x}_e$ is CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$) for (1) if and only if $\boldsymbol{y}_e=\boldsymbol{0}$ is CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$) for (9). The components g_i^1, $i=\overline{1,m}$, and g_j^2, $j=\overline{1,n}$, of the activation functions $\boldsymbol{g}^1$ and, respectively, $\boldsymbol{g}^2$, are Lipschitz continuous and satisfy the following conditions derived from (2):

$$(11)\quad 0\le\left|g_i^1(\xi)\right|\le L_i^1|\xi|,\ \ 0\le\left|g_j^2(\xi)\right|\le L_j^2|\xi|,$$

for all $\xi\in\mathbb{R}$.

Assuming that for an arbitrary $c>0$, the solution to (9) initiated so that $\left|\boldsymbol{y}^k(t_0)\right|\le c\boldsymbol{p}^k(t_0)$, $k=\overline{1,2}$, satisfies $\left|\boldsymbol{y}^k(t)\right|\le c\boldsymbol{p}^k(t)$, $k=\overline{1,2}$, for a certain $t\ge t_0$, and taking (11) into account, we get

$$\begin{aligned}
&\left|y_i^1(t+1)\right|\le\left|a_i^1\right|\left|y_i^1(t)\right|+\sum_{j=1}^{n}\left|w_{ij}^2\right|\left|g_j^2\left(y_j^2(t)\right)\right|\le\\
&\le\max\left\{|\underline{a}_i^1|,|\overline{a}_i^1|\right\}\left|y_i^1(t)\right|+\sum_{j=1}^{n}\max\left\{|\underline{w}_{ij}^2|,|\overline{w}_{ij}^2|\right\}L_j^2\left|y_j^2(t)\right|\le\\
&\le c\left[\hat{a}_i^1p_i^1(t)+\sum_{j=1}^{n}\hat{w}_{ij}^2L_j^2p_j^2(t)\right],\ i=\overline{1,m},\\
&\left|y_j^2(t+1)\right|\le\left|a_j^2\right|\left|y_j^2(t)\right|+\sum_{i=1}^{m}\left|w_{ji}^1\right|\left|g_i^1\left(y_i^1(t)\right)\right|\le\\
&\le\max\left\{|\underline{a}_j^2|,|\overline{a}_j^2|\right\}\left|y_j^2(t)\right|+\sum_{j=1}^{n}\max\left\{|\underline{w}_{ji}^1|,|\overline{w}_{ji}^1|\right\}L_i^1\left|y_i^1(t)\right|\le\\
(12)\quad&\le c\left[\hat{a}_j^2p_j^2(t)+\sum_{j=1}^{n}\hat{w}_{ji}^1L_i^1p_i^1(t)\right],\ j=\overline{1,n}.
\end{aligned}$$

If (7) is satisfied, it follows that $\left|\boldsymbol{y}^k(t+1)\right|\le c\boldsymbol{p}^k(t+1)$, $k=\overline{1,2}$; through mathematical induction the fulfillment of $\left|\boldsymbol{y}^k(t)\right|\le c\boldsymbol{p}^k(t)$, $k=\overline{1,2}$, is ensured for all $t\ge t_0$, meaning that $\boldsymbol{y}_e=\boldsymbol{0}$ is CWAS($c\boldsymbol{p}^1,c\boldsymbol{p}^2$) for (9). Since this happens for all $c>0$, the equilibrium point $\boldsymbol{x}_e$ of BAM (1) is GCWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$), showing that BAM (1) is CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$). This conclusion can be drawn for all BAMs described by (1) with $\boldsymbol{A}^k\in\mathcal{A}^k$, $\boldsymbol{W}^k\in\mathcal{W}^k$, $k=\overline{1,2}$, which completes the proof.

Remark 3. Let us introduce the augmented vector function

$$\boldsymbol{p}:\mathbb{Z}_+\to\mathbb{R}^{m+n},\ \boldsymbol{p}(t)=\left[\boldsymbol{p}^1(t)^\tau\ \ \boldsymbol{p}^2(t)^\tau\right]^\tau,\qquad(13)$$

and the matrix $\Theta\in\mathbb{R}^{(m+n)\times(m+n)}$ defined by

$$(14)\quad \begin{aligned}\Theta&=\begin{bmatrix}\hat{\boldsymbol{A}}^1&\boldsymbol{O}\\\boldsymbol{O}&\hat{\boldsymbol{A}}^2\end{bmatrix}+\begin{bmatrix}\boldsymbol{O}&\hat{\boldsymbol{W}}^2\\\hat{\boldsymbol{W}}^1&\boldsymbol{O}\end{bmatrix}\begin{bmatrix}\boldsymbol{L}^1&\boldsymbol{O}\\\boldsymbol{O}&\boldsymbol{L}^2\end{bmatrix}=\\&=\begin{bmatrix}\hat{\boldsymbol{A}}^1&\hat{\boldsymbol{W}}^2\boldsymbol{L}^2\\\hat{\boldsymbol{W}}^1\boldsymbol{L}^1&\hat{\boldsymbol{A}}^2\end{bmatrix},\end{aligned}$$

where matrices $\hat{\boldsymbol{A}}^k$, $\hat{\boldsymbol{W}}^k$, $\boldsymbol{L}^k$, $k=\overline{1,2}$, given by (8). The sufficient condition (7) stated by Theorem 1 can be equivalently written as

$$(15)\quad \boldsymbol{p}(t+1)\ge\Theta\boldsymbol{p}(t),\ \forall t\in\mathbb{Z}_+,$$

Remark 3 suggests us to explore the role of matrix Θ (14) in ensuring the CWAS($\boldsymbol{p}^1,\boldsymbol{p}^2$) of IBAM($\mathcal{A}^1,\mathcal{A}^2,\mathcal{W}^1,\mathcal{W}^2$). Let us first notice the special structure of the test matrix Θ, which is *nonnegative* (all its elements are nonnegative). This remark motivates us to present some preparatory results.

Lemma 1. Let $\Theta=[\theta_{rs}]\in\mathbb{R}^{q\times q}$ be a nonnegative square matrix and let us denote by $\lambda_r(\Theta)$, $r=\overline{1,q}$, its eigenvalues. Then, Θ has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(\Theta)$, which fulfils the dominance condition $|\lambda_r(\Theta)|\le\lambda_{\max}(\Theta)$ for all $r=\overline{1,q}$. Moreover, $\theta_{rr}\le\lambda_{\max}(\Theta)$, $r=\overline{1,q}$.
Proof: It results from (Pastravanu and Voicu, 2002), Lemma 2.1, and from (Horn and Johnson, 1985), Corollary 8.1.20.

Lemma 2: If $\Theta,\Psi\in\mathbb{R}^{q\times q}$ are nonnegative matrices satisfying $\Theta\le\Psi$, then $\lambda_{\max}(\Theta)\le\lambda_{\max}(\Psi)$.
Proof: It results from (Horn and Johnson, 1985), Theorem 8.1.18.

Lemma 3: If $\Theta=[\theta_{rs}]\in\mathbb{R}^{q\times q}$ is a nonnegative matrix, then, for any $\sigma\in\mathbb{R}_+$, with $\lambda_{\max}(\Theta)<\sigma$, there exists a positive vector $\gamma\in\mathbb{R}^q,\gamma>\boldsymbol{0}$, such that $\Theta\gamma<\sigma\gamma$.
Proof: Since Θ is nonnegative, for any $\sigma\in\mathbb{R}_+$, $\lambda_{\max}(\Theta)<\sigma$, there exists an $\varepsilon=\varepsilon(\sigma)>0$ such that $\lambda_{\max}(\Theta+\varepsilon\boldsymbol{E})\le\sigma$, where $\boldsymbol{E}=\left[e_{rs}\right]\in\mathbb{R}^{q\times q}$, with $e_{rs}=1$, $r,s=\overline{1,q}$. Thus, for the Perron eigenvector $\gamma\in\mathbb{R}^q$, $\gamma>\boldsymbol{0}$, of the positive matrix $\Theta+\varepsilon\boldsymbol{E}>\boldsymbol{O}$ we can write $\Theta\gamma<(\Theta+\varepsilon\boldsymbol{E})\gamma=\lambda_{\max}(\Theta+\varepsilon\boldsymbol{E})\gamma\le$ $\le\sigma\gamma$. Note that when Θ is irreducible, the existence of the Perron-Frobenius eigenvector $\gamma\in\mathbb{R}^q$, $\gamma>\boldsymbol{0}$, ensures the equality $\Theta\gamma=\lambda_{\max}(\Theta)\gamma$.

We are now able to establish the following result.

Theorem 2. If matrix Θ defined by (14) is Schur stable, then there exist two vector functions $\boldsymbol{p}^k(t)$, $k=\overline{1,2}$, satisfying the conditions from Definition 1 so that IBAM($\mathcal{A}^1, \mathcal{A}^2, \mathcal{W}^1, \mathcal{W}^2$) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$).

Proof: Indeed, if Θ is Schur stable, then the vector function

$$(16) \quad \boldsymbol{p}(t) = \Theta^t \boldsymbol{p}(0) + \sum_{\zeta=0}^{t-1} \Theta^{\zeta} \boldsymbol{v}(t-1-\zeta)$$

(defined with $\boldsymbol{p}(0) > \boldsymbol{0}$ and adequate $\boldsymbol{v}(\zeta) \geq 0$, $\zeta \in \mathbb{Z}_+$, such that $\lim_{t\to\infty} \sum_{\zeta=0}^{t} \Theta^{\zeta} \boldsymbol{v}(t-\zeta) = \boldsymbol{0}$), satisfies the algebraic inequality (15) and $\boldsymbol{p}(t) > \boldsymbol{0}$, $\forall t \in \mathbb{Z}_+$, $\lim_{t\to\infty} \boldsymbol{p}(t) = \boldsymbol{0}$. The two functions ensuring CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) result from the appropriate partitioning of $\boldsymbol{p}$, in accordance to (13).

3.2. CWEAS of IBAMs

Theorem 3. IBAM($\mathcal{A}^1, \mathcal{A}^2, \mathcal{W}^1, \mathcal{W}^2$) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$) if the following algebraic inequalities hold

$$(17) \quad \begin{aligned} \sigma\boldsymbol{\alpha}^1 &\geq \hat{\boldsymbol{A}}^1\boldsymbol{\alpha}^1 + \hat{\boldsymbol{W}}^2\boldsymbol{L}^2\boldsymbol{\alpha}^2, \\ \sigma\boldsymbol{\alpha}^2 &\geq \hat{\boldsymbol{A}}^2\boldsymbol{\alpha}^2 + \hat{\boldsymbol{W}}^1\boldsymbol{L}^1\boldsymbol{\alpha}^1. \end{aligned}$$

Proof: It is a direct consequence of Theorem 1 when the time-dependence of vector-functions $\boldsymbol{p}^k(t)$, $k=\overline{1,2}$, is given by (6).

Remark 4. The same as in Remark 3, let us notice that by introducing the augmented vector $\boldsymbol{\alpha} = \left[\boldsymbol{\alpha}^{1\tau} \ \boldsymbol{\alpha}^{2\tau}\right]^{\tau} \in \mathbb{R}^{m+n}$, inequalities (17) may be written in the equivalent matrix-form

$$(18) \quad \sigma\boldsymbol{\alpha} \geq \Theta\boldsymbol{\alpha},$$

where Θ is given by (14).

Similarly to Theorem 2, the following result is available for CWEAS.

Theorem 4. If matrix Θ defined by (14) is Schur stable, then there exist two positive vectors $\boldsymbol{\alpha}^1 \in \mathbb{R}^m, \boldsymbol{\alpha}^1 > \boldsymbol{0}$, $\boldsymbol{\alpha}^2 \in \mathbb{R}^n, \boldsymbol{\alpha}^2 > \boldsymbol{0}$, and a scalar $\sigma \in (0,1)$, so that IBAM($\mathcal{A}^1, \mathcal{A}^2, \mathcal{W}^1, \mathcal{W}^2$) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$).

Proof: If matrix Θ is Schur stable, then Lemma 3 ensures the existence of $\sigma \in \mathbb{R}_+$, $\lambda_{\max}(\Theta) < \sigma < 1$ and $\boldsymbol{\alpha} \in \mathbb{R}^{m+n}, \boldsymbol{\alpha} > 0$, satisfying inequality (18). The two positive vectors ensuring CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$) result from the appropriate partitioning of $\boldsymbol{\alpha}$.

3.3. CWAS / CWEAS of a BAM

The generality of our results on CWAS/CWEAS of IBAMs includes the particular case of a BAM with fixed parameters (i.e. without uncertainties), obtained for $\underline{\boldsymbol{A}}^k = \overline{\boldsymbol{A}}^k = \boldsymbol{A}^k$, $\underline{\boldsymbol{W}}^k = \overline{\boldsymbol{W}}^k = \boldsymbol{W}^k$, $k=\overline{1,2}$, in (3). The CWAS/CWEAS approach relies on the replacement of the test matrix $\Theta \in \mathbb{R}^{(m+n)\times(m+n)}$, built according to (14), by the test matrix $\boldsymbol{\Omega} \in \mathbb{R}^{(m+n)\times(m+n)}$ defined as:

$$(19) \quad \begin{aligned} \boldsymbol{\Omega} &= \begin{bmatrix} |\boldsymbol{A}^1| & \boldsymbol{O} \\ \boldsymbol{O} & |\boldsymbol{A}^2| \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} & |\boldsymbol{W}^2| \\ |\boldsymbol{W}^1| & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{L}^1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}^2 \end{bmatrix} = \\ &= \begin{bmatrix} |\boldsymbol{A}^1| & |\boldsymbol{W}^2|\boldsymbol{L}^2 \\ |\boldsymbol{W}^1|\boldsymbol{L}^1 & |\boldsymbol{A}^2| \end{bmatrix}, \end{aligned}$$

with matrices $\boldsymbol{L}^k$, $k=\overline{1,2}$, given by (8).

Theorem 5. If matrix $\boldsymbol{\Omega}$ defined by (19) is Schur stable, then

(**a**) BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) if the augmented vector function $\boldsymbol{p}(t)$ given by (13) fulfills the difference inequality

$$(20) \quad \boldsymbol{p}(t+1) \geq \boldsymbol{\Omega}\, \boldsymbol{p}(t), \ \forall t \in \mathbb{Z}_+;$$

(**b**) BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$) if the scalar σ and the vector $\boldsymbol{\alpha} = \left[\boldsymbol{\alpha}^{1\tau} \ \boldsymbol{\alpha}^{2\tau}\right]^{\tau} \in \mathbb{R}^{m+n}$ fulfill the algebraic inequality

$$(21) \quad \sigma\boldsymbol{\alpha} \geq \boldsymbol{\Omega}\boldsymbol{\alpha}.$$

Remark 5. The conditions $|a_i^1| < 1$, $i=\overline{1,m}$, $|a_j^2| < 1$, $j=\overline{1,n}$, are necessary for matrix $\boldsymbol{\Omega}$ defined by (19) to be Schur stable. It is worth noticing that such conditions are formulated as working hypotheses in most papers dealing with discrete-time BAMs.

Remark 6. If BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), then, for its unique equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau} \ \boldsymbol{x}_e^{2\tau}\right]^{\tau}$ we can write

$$\forall \varepsilon > 0,\ \forall t_0 \in \mathbb{Z}_+,\ \forall \boldsymbol{x}_0 \in \mathbb{R}^{m+n},$$
$$(22)\quad \|\boldsymbol{x}_0 - \boldsymbol{x}_e\|_\infty^{\mathrm{A}} \le \varepsilon \Rightarrow \|\boldsymbol{x}(t;t_0,\boldsymbol{x}_0) - \boldsymbol{x}_e\|_\infty^{\mathrm{A}} \le \varepsilon\sigma^{t-t_0},$$
$$\forall t \in \mathbb{Z}_+,\ t \ge t_0,$$

where the vector norm $\| \ \|_\infty^{\mathrm{A}}$ is defined by $\|\boldsymbol{x}\|_\infty^{\mathrm{A}} = \|\mathrm{A}\boldsymbol{x}\|_\infty$, $\forall \boldsymbol{x} \in \mathbb{R}^{m+n}$, with

$$(23)\quad \mathrm{A} = \mathrm{diag}\left\{1/\alpha_1^1, \ldots, 1/\alpha_m^1, 1/\alpha_1^2, \ldots, 1/\alpha_n^2\right\}.$$

This shows that for each concrete neural network the definition of exponential stability of $\boldsymbol{x}_e$, e.g. (Michel and Wang, 1995), pp.107, with respect to the norm $\| \ \|_\infty^{\mathrm{A}}$ is fulfilled in the particular case $\delta(\varepsilon) = \varepsilon$, $\forall \varepsilon > 0$.

Similarly, the definition of global exponential stability of $\boldsymbol{x}_e$, e.g. (Michel and Wang, 1995), pp.108, is satisfied for the particular case $M = 1$,

$$(24)\quad \begin{array}{l} \forall t_0 \in \mathbb{Z}_+, \forall \boldsymbol{x}_0 \in \mathbb{R}^{m+n}, \\ \|\boldsymbol{x}(t;t_0,\boldsymbol{x}_0) - \boldsymbol{x}_e\|_\infty^{\mathrm{A}} \le M\sigma^{t-t_0} \|\boldsymbol{x}_0 - \boldsymbol{x}_e\|_\infty^{\mathrm{A}},\ \forall t \ge t_0. \end{array}$$

Remark 7. In terms of the matrix norms induced by the vector norms, the sufficient condition for BAM (1) to be CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$) can be formulated as

$$(25)\quad \|\Omega\|_\infty^{\mathrm{A}} = \left\|\mathrm{A}\Omega\mathrm{A}^{-1}\right\|_\infty \le \sigma < 1,$$

where matrix A is given by (23). This is a direct consequence of Theorem 5(b).

5. CONCLUSIONS

This paper provides easy-to-apply algebraic criteria for exploring the componentwise (exponential) asymptotic stability of discrete-time BAM neural networks with interval type parameters. These criteria are formulated in terms of Schur stability of a test matrix adequately built from the nonlinear system. Theorems 2 and 4 are the key elements of our approach that gives a qualitative characterization of the dynamics at the level of the state vector components. This novel point of view refines the classical results in stability theory based on global information about the state vector, expressed in terms of arbitrary norms.

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