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ON THE STABILITY OF THE CELLULAR NEURAL NETWORKS WITH TIME LAGS

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Abstract: Cellular neural networks (CNNs) are recurrent artificial neural networks. Due to their cyclic connections and to the neurons' nonlinear activation functions, recurrent neural networks are nonlinear dynamic systems, which display stable and unstable fixed points, limit cycles and chaotic behavior. Since the field of neural networks is still a young one, improving the stability conditions for such systems is an obvious and quasi-permanent task. This paper focuses on CNNs affected by time delays. We are interested to obtain sufficient conditions for the asymptotical stability of a cellular neural network with time delay feedback and zero control templates. For this purpose we shall use a method suggested by Malkin (1952), where the "exact" Liapunov-Krasovskii functional will be constructed according the procedure proposed by Kharitonov (2001) for stability analysis of uncertain linear time delay systems.

Keywords: cellular neural network, time delays, "exact" Liapunov-Krasovskii functional, absolute stability

1. INTRODUCTION

Cellular neural networks (CNNs), introduced in 1988 (Chua and Yang, 1988), are artificial recurrent neural networks displaying a multidimensional array of cells and local interconnections among the cells. CNNs have been successfully applied to signal and image processing, shape extraction and edge detection. In such applications stability and other problems of dynamical behaviour of the CNN are equally important. These properties are necessary for the network to achieve its goal and have to be checked on the mathematical model.

In the last ten years the research was oriented towards the dynamics of the networks affected by time delays due to the signal propagation at the synapses level of the biologic brain or the reacting lag in the case of the artificial neural network. These lags may introduce oscillations or may lead to instability of the network. We are interested to obtain sufficient conditions for the asymptotical stability of a cellular neural network with time delay feedback and zero control templates. For this purpose we shall use a method suggested by Malkin (Malkin, 1952; see also Barbashin, 1970), where the "exact" Liapunov-Krasovskii functional will be constructed according the procedure proposed by Kharitonov (2001) for stability analysis of uncertain linear time delay systems. These conditions are independent of the delay parameter.

2. THE MATHEMATICAL MODEL AND PROBLEM STATEMENT

Consider a cellular neural network with time delay feedback and zero control templates

$$(1)\quad \dot{z}_i = -a_i z_i(t) + \sum_{j\in N} c_{ij} g_j\big(z_j(t-\tau_j)\big) + I_i$$

where j is the index for the cells of the nearest neighborhood N of the i^{th} cell, a_i is a positive

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parameter, c_{ij} are synaptic weights (which can have an inhibitory effect if $c_{ij} < 0$, or an excitatory one if $c_{ij} > 0$), I_i is the bias and τ_j are positive delays.

The nonlinearities for the cellular neural networks are of the bipolar ramp type:

$$(2) \quad g_i(z_i) = \tfrac{1}{2}\left(|z_i + 1| - |z_i - 1|\right)$$

what means they are bounded, monotonically increasing and globally Lipschitzian functions, with the Lipschitz constant $L_i = 1$.

Without loss of the generality, using a change of the coordinates, $x_i = z_i - z_i^*$, one can shift the equilibrium point z^* to the origin so that system (1) can be written into the form:

$$(3) \quad \dot{x}_i(t) = -a_i x_i(t) + \sum_{j \in N} c_{ij} f_j\left(x_j(t - \tau_j)\right), \; i = \overline{1,n}$$

Using a method proposed by Malkin (1952), we assume that there exists $k_i > 0$ such that the nonlinearities satisfy

$$(4) \quad 0 \le k_i - \underline{k_i} < \frac{f_i(\sigma)}{\sigma} < k_i + \overline{k_i}$$

and that for $f_i(x_i) = k_i x_i$ the system

$$(5) \quad \dot{x}_i(t) = -a_i x_i(t) + \sum_{j \in N} c_{ij} k_j x_j(t - \tau_j), \; i = \overline{1,n}$$

is exponentially stable. We underline that (5) is a normal condition taking into account the properties of the activation functions of CNNs neurons.

Denoting

$$(6) \quad A_0 = diag\left(-a_i\right)_1^n$$

$$(7) \quad C_j = \begin{pmatrix} 0 & \dots & 0 & c_{1j} & 0 & \dots & 0 \\ 0 & \dots & 0 & c_{2j} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & c_{nj} & 0 & \dots & 0 \end{pmatrix}$$

$$(8) \quad A_j = C_j \cdot \underset{i}{diag}\left(k_i\right)_1^n$$

system (5) may be written into the form

$$(9) \quad \dot{x}(t) = A_0 x(t) + \sum_{j=1}^{n} A_j x\left(t - \tau_j\right)$$

with the initial condition $x_i(\theta) = \varphi(\theta)$, for $\theta \in [-\tau, 0]$, where $\tau = \max_j \tau_j$, $\varphi \in \boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$.

Consider now the perturbed system:

$$(10) \quad \dot{y}(t) = A_0 y(t) + \sum_{j=1}^{n} \left(C_j \cdot \underset{i}{diag}\left(k_i + b_i\right)_1^n \right) y\left(t - \tau_j\right)$$

which can be written as

$$(11) \quad \dot{y}(t) = A_0 y(t) + \sum_{j=1}^{n} \left(A_j + \Delta_j\right) y\left(t - \tau_j\right)$$

with

$$(12) \quad \Delta_j = C_j \cdot \underset{i}{diag}\left(b_i\right)_1^n, \; j = \overline{1,n}$$

We are interested to find conditions so that the perturbed system (11) remains exponentially stable for all $b_i \in \left(-\underline{k_i}, \overline{k_i}\right), \forall i = \overline{1,n}$. In fact, $\left(-\underline{k_i}, \overline{k_i}\right)$ represents the proper interval for the nonlinear functions f_i attached to each cell of the network. In this way we will find sufficient conditions for the exponentially stability of the nonlinear system (3).

3. MAIN RESULT

Given positive definite n x n matrices P_0, P_j, R_j, $j = \overline{1,n}$ let us define on $\boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$ the positive definite functional

$$(13) \quad \begin{aligned} W\left(\phi(\cdot)\right) &= \phi^T(0) P_0 \phi(0) + \sum_{j=1}^{n} \phi^T(-\tau_j) P_j \phi(\tau_j) \\ &+ \sum_{j=1}^{n} \int_{-\tau_j}^{0} \phi^T(\theta) R_j \phi(\theta) d\theta \end{aligned}$$

Since system (9) is exponentially stable, there exists a Liapunov-Krasovskii functional $V(\phi(\cdot))$ so that along the solutions of (9) we have the equality

$$(14) \quad \frac{d}{dt} V(x(t + \cdot)) = -W(x(t + \cdot))$$

The "exact" Liapunov-Krasovskii functional is of the form

$$(15) \quad \begin{aligned} & V(x(t+\cdot)) = x^T(t) U(0) x(t) \\ & + \sum_{j=1}^{n} 2x^T(t) \int_{-\tau_i}^{0} U(-\tau_i - \theta) C_j \left(\underset{i}{diag}(k_i)_1^n \right) x(t+\theta) d\theta \\ & + \sum_{k=1}^{n} \sum_{j=1}^{n} \int_{-\tau_k}^{0} x^T(t+\theta_2) \left(\underset{i}{diag}(k_i)_1^n \right) C_j \cdot \\ & \cdot \left(\int_{-\tau_i}^{0} U(\theta_1 - \theta_2 + \tau_k + \tau_j) C_j \left(\underset{i}{diag}(k_i)_1^n \right) d\theta_1 \right) d\theta_2 \\ & + \sum_{j=1}^{n} \int_{-\tau_i}^{0} x^T(t+\theta) \left[(\tau_j + \theta) R_j + P_j \right] x(t+\theta) d\theta \end{aligned}$$

where, since the system (9) is exponentially stable, the matrix valued function

$$(16)\quad U(\tau)=\int_0^{\infty} K^T(t)\left[P_0+\sum_{j=1}^{n}\left(P_j+\tau_j R_j\right)\right]K(t+\tau)dt$$

is well defined for all $\tau \in \boldsymbol{R}$; here $K(t)$ is the fundamental matrix associated to the system (9) (see Kharitonov, 2001).

Following the steps in (Kharitonov, 2001) the time derivative of Liapunov-Krasovskii functional along the solutions of the perturbed system (11) is

$$(17)\quad \begin{aligned}&\frac{d}{dt}V(y(t+\cdot))=-W(y(t+\cdot))+2\left[\sum_{j=1}^{n}\Delta_j y(t+\tau_j)\right]^T\cdot\\ &\cdot\left[U(0)y(t)+\sum_{j=1}^{n}\int_{-\tau_j}^{0}U^T(\tau_j+\theta)A_j y(t+\theta)d\theta\right]\end{aligned}$$

We assume that b_i, $i=\overline{1,n}$ are so that the matrices Δ_j, defined by (12), are constant and satisfy the condition

$$(18)\quad \Delta_j^T H_j \Delta_j \le \rho_j I,\ j=\overline{1,n}$$

where H_j are definite positive matrices, ρ_j are given positive numbers and I is the identity matrix.

For the derivative of the functional (15) along the trajectories of the perturbed system (11) one obtains the following upper bound:

$$(19)\quad \begin{aligned}&\frac{d}{dt}V(y(t+\cdot))\le -y^T(t)\left[P_0-\mu U^T(0)\sum_{j=1}^{n}H_j^{-1}U(0)\right]y(t)\\ &-\sum_{j=1}^{n}\int_{-\tau_j}^{0}y^T(t+\theta)\left[R_j-\mu A_j^T U(\tau_j+\theta)\cdot\right.\\ &\left.\cdot\left(\sum_{k=1}^{n}H_k^{-1}\right)U^T(\tau_j+\theta)A_j\right]y(t+\theta)d\theta\\ &-\sum_{j=1}^{n}y^T(t-\tau_j)\left[P_j-\frac{2}{\mu}\rho_j I\right]y(t-\tau_j)\end{aligned}$$

where is assumed that $H_j := \frac{1}{\mu}H_j$, $j=\overline{0,n}$.

From the above inequality one derives the following theorem:

Theorem: *Let system (5) be exponentially stable. Then system (3) is exponentially stable for all nonlinearities of the form* $f_i(x_i)=\left(k_i+b_i(x_i)\right)x_i$, *with* $b_i(x_i)\in\left(-\underline{k_i},\overline{k_i}\right),\forall i=\overline{1,n}$ *defined by (12), if there exist definite positive matrices* P_0, P_j, R_j, $j=\overline{1,n}$ *and a positive value* μ, *such that*

$$P_0>\mu U^T(0)\left(\sum_{j=1}^{n}H_j^{-1}\right)U(0)$$

$$R_j>\mu A_j^T U(\tau_j+\theta)\left(\sum_{k=1}^{n}H_k^{-1}\right)U^T(\tau_j+\theta)A_j$$

$$P_j>\frac{2}{\mu}\rho_j I\ .$$

Here b_i is a nonlinear function whose form is derived from (4):

$$(20)\quad b_i(x_i)=\frac{f_i(x_i)}{x_i}-k_i$$

The Malkin's method idea is: if the Liapunov function(al) and its derivative - both being quadratic forms - have good sign properties for all $b_i\in\left(-\underline{k_i},\overline{k_i}\right)$, then for any fixed $x_i\ne 0$ one can obtain b_i from (20) and for $b_i(x_i)\in\left(-\underline{k_i},\overline{k_i}\right)$ the properties of the Liapunov function(al) do not change.

Remark: *If the theorem conditions are satisfied, then terms* b_i *may be even time varying within the interval* $\left(-\underline{k_i},\overline{k_i}\right)$, *if they are integrable functions of t.*

Indeed, for some constants $\varepsilon>0$ and $\gamma>0$ the functional $V(\varphi)$ verifies the inequalities

$$(21)\quad \varepsilon\|\varphi(0)\|^2\le V(\varphi)\le\gamma\|\varphi\|_h^2$$

where $\|\varphi\|=\max\limits_{-\tau_i\le s\le 0}\|\varphi(s)\|$.

The strictly positive lower bound of V (with $\varepsilon>0$) is not valid in general since $V(\varphi)$ is a positive definite quadratic functional on an infinite dimensional space and the spectrum of a positive operator does not meet in general the compacity assumption. In our case, however taking into account (15), the well delimitation of V from 0 i.e. $\varepsilon>0$ is secured.

Along the solutions of the perturbed system (11) the derivative of the functional is

$$(22)\quad \begin{aligned}&\frac{d}{dt}V(y(t+\cdot))=-\widetilde{W}(y(t+\cdot))\\ &=-W(y(t+\cdot))+2\left(\sum_{j=1}^{n}\Delta_j y(t-\tau_j)\right)^T\cdot\\ &\cdot\left[U(0)y(t)+\sum_{j=1}^{n}\int_{-\tau_i}^{0}U^T(\tau_j+\theta)A_j y(t+\theta)d\theta\right]\end{aligned}$$

If matrices Δ_j, $j=\overline{1,n}$ depend on t and (or) on $y(t-\tau_j)$, $j=\overline{1,n}$ but they satisfy inequalities (12), then there exists $\varepsilon > 0$ such that

$$\widetilde{W}(\varphi) \geq \varepsilon W(\varphi) \quad (23)$$

Here also the lower bound on $\widetilde{W}$ may not be strictly positive in general but, if one takes into account (19) the delimitation of $\widetilde{W}$ from 0 is again secured.

Conditions (21) and (23) imply that the zero solution of the perturbed system (11) is global asymptotically stable.

4. CONCLUDING REMARKS

The present paper extend our preview work (Danciu and Răsvan 2001; Danciu, 2003, 2004), and state sufficient condition for the exponentially stability of a cellular neural network with time delay feedback and zero control templates. The result is based mainly on a method suggested by Malkin (1952) for studying the nonlinear systems via its linearisations, where the “exact” Liapunov-Krasovskii functional was constructed according the procedure proposed by Kharitonov (2001) for stability analysis of uncertain linear time delay systems.

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