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ASPECTS REGARDING THE INFLUENCE OF THE DAMPING OVER THE RUNNING OF A THERAPEUTIC APPARATUS

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ABSTRACT

The paper studies a therapeutic apparatus from the point of view of the influence of damping over its functioning, for various working conditions. The apparatus is intended for the subjects suffering from severe pains, taking into account that the back pain is the affection of the 21st century, and it aims to provide an objective reference for the progress and the result of the treatment, useful for both the physician and for the subject.

KEYWORDS: medical apparatus, electromechanical device, biomechanics, mechanical vibrations

1. GENERAL NOTIONS

Statistical data show that approximately seven out of ten persons suffer from back pains once or several times in a lifetime, while 7% of the population is affected daily by the same disease.

Scientific research highlighted that movement acts as a clinical treatment for painful joints, so that bed rest is no more necessary for such affections.

As a consequence, taking into account both medical and economical reasons – eliminating or reducing pain and fast recovery for normal activity – a set of medical concepts and the corresponding apparatuses were developed. Some of these apparatuses are based on the system of active medical recovery practiced against the deconditioning syndrome.

Such therapeutical techniques through motion allow for the balanced recovery of all major muscle groups, the recovery of the normal coordination by practicing the movements in a controlled manner, improvement of the metabolism, the fortification of muscles and of the spinal column and the relaxation of the muscle mass.

The medical recovery system for the spinal column treats non-specific back pains, degenerative modifications, herniated discs and post-operative situations, by recovery programs specific to each subject.

By means of such apparatuses, the force and the mobility of the backbone are tested in all possible movement planes, while the results are stored, in order to achieve a relevant individual profile and to compare this profile with predictive data.

The characteristics of the damping system of the apparatus have an important role for the success of the treatment and for the evaluation performed during the program.

2. PRESENTATION OF THE STUDY MODEL

The model of such a therapeutical device, chosen to analyze the influence of damping [5], is illustrated in Figure 1. The device allows to stabilize the pelvis, so that the exercises should be performed in safe conditions, without pains, the movement being controlled and allowing to isolate the target muscle groups.

The device consists in the following elements:

1. the case of the device;
2. elastic bar, hinged in point A and fixed in point B, of length $3a$;
3. spring system with the equivalent elasticity constant k_1;

4. damper, whose damping coefficient c_1 includes the characteristic of element 1;
5. adjustable mass m_1;
6. spring system with the equivalent elasticity constant k_2;
7. damper with the damping coefficient c_2;
8. adjustable mass m_2;
9. rigid transmission bar, hinged in point C, of length $3a$;
10. torsion spring of elasticity constant k_t;
11. spring system with the equivalent elasticity constant k_3;
12. damper with the damping coefficient c_3;
13. transmission system of the periodical force $F(t)$, fixed on the element 9 and on the damping system 11,12 in point D, as well as on the plate 14 in point E;
14. rigid transmission plate;
15. traction rings;
16. swinging support of the body, which allows for the motion in sitting, as well as in standing position.

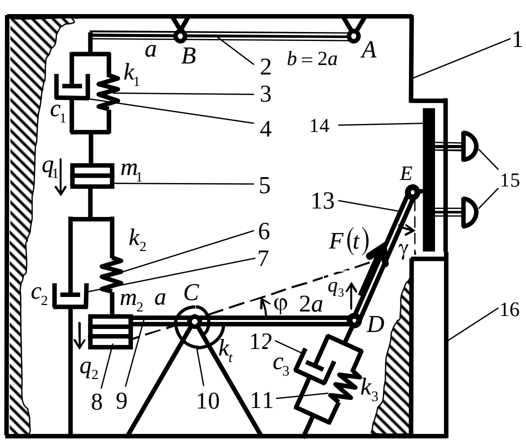

Fig. 1. Study model

The motion is produced by acting the rings 15 and is transmitted to the system through elements 14 and 13. By varying the sizes of masses m_1 and m_2, the direction angle γ, the position of the body, as well as the type of force $F(t)$ used, the apparatus allows, under the physician’s prescription, to solve medical recovery problems or to perform conservation exercises.

3. DIFFERENTIAL EQUATIONS OF MOTION OF THE SYSTEM

The presented mechanical model has two degrees of freedom. Displacements q_1 and q_2 were chosen as independent generalized coordinates. The third displacement is linked to the second one by the kinematical relation [3]

$$q_3 = 2a\,\varphi = 2a\frac{q_2}{a} = 2q_2 . \qquad (1)$$

The elasticity constant of the bar 2 is:

$$k_{BS} = \frac{3EI}{(a+b)\cdot b^2} = \frac{EI}{a^3} . \qquad (2)$$

The equivalent elasticity constant for the bar 2 and for the spring 1, considered as elastic elements connected in series, is:

$$k_{ech} = \frac{1}{\dfrac{1}{k_{BS}} + \dfrac{1}{k_1}} = \frac{EIk_1}{k_1 a^3 + EI} . \qquad (3)$$

The differential equations of the small oscillations of the system can be determined by using the formalism of the Lagrange equations of the second species [1],

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E}{\partial \dot q_1}\right) - \dfrac{\partial E}{\partial q_1} = -\dfrac{\partial E_p}{\partial q_1} - \dfrac{\partial D}{\partial \dot q_1} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E}{\partial \dot q_2}\right) - \dfrac{\partial E}{\partial q_2} = -\dfrac{\partial E_p}{\partial q_2} - \dfrac{\partial D}{\partial \dot q_2} + Q(t), \end{cases} \qquad (4)$$

where the following notations were introduced [1]:

– the kinetic energy of the system,

$$E = \frac{1}{2} m_1 \dot q_1^2 + \frac{1}{2} m_2 \dot q_2^2 , \qquad (5)$$

– the potential energy of the system,

$$E_p = \frac{1}{2}(k_{ech} + k_2) q_1^2 + \frac{1}{2}\left(k_2 + 4k_3\cos^2\gamma + \frac{k_t}{a^2}\right) q_2^2 - k_2 q_1 q_2 , \qquad (6)$$

– the Rayleigh dissipation function

$$D = \frac{1}{2} c_{ech} \dot q_1^2 + \frac{3}{2} c_3 \dot q_2^2 \qquad (7)$$

– the generalized perturbation force

$$Q(t) = F(t)\cos\gamma . \qquad (8)$$

By calculating and replacing the derivatives in system (4), the differential equations of the small oscillations of the system result (considering $\gamma = 30^0$):

$$\begin{cases} m_1\ddot{q}_1 + c_{ech}\dot{q}_1 + (k_{ech} + k_2)q_1 - k_2 q_2 = 0 \\ m_2\ddot{q}_2 + 3c_3\dot{q}_2 - k_2 q_1 + \\ \quad + \left(k_2 + 3k_3 + \dfrac{k_t}{a^2}\right)q_2 = Q(t). \end{cases} \tag{9}$$

$$Q(t) = F(t)\frac{\sqrt{3}}{2}. \tag{10}$$

By choosing the following expressions for the parameters of the apparatus,

$$\begin{cases} \dfrac{EI}{a^3} = 4k, \quad k_1 = 4k, \quad k_2 = 6k \\ k_3 = 4k, \quad k_t = 4a^2k \\ c_{BS} = 4c, \quad c_{12} = 2c, \quad c_3 = 8c \\ m_1 = m, \quad m_2 = 4m, \end{cases} \tag{11}$$

system (9) leads to the decoupled equation

$$4m^2(\ddot{q}_2 - \ddot{q}_1) + 24mc(\dot{q}_2 - \dot{q}_1) + \\ + 38mk(q_2 - q_1) = mQ(t), \tag{12}$$

which can be rewritten in the form

$$\ddot{q}_{dif} + 2\alpha\dot{q}_{dif} + \omega^2 q_{dif} = f(t), \tag{13}$$

where

$$q_{dif} = q_2 - q_1, \tag{14}$$

$$\alpha = 3\frac{c}{m}, \quad \omega^2 = \frac{19}{2}\cdot\frac{k}{m}, \quad f(t) = \frac{Q(t)}{m}. \tag{15}$$

The differential equation (13) will be integrated with the initial conditions

$$q_{dif}(t=0) = 0, \quad \dot{q}_{dif}(t=0) = 0. \tag{16}$$

In the following, two forms of the perturbating force will be analyzed.

4. CONSTANT PERTURBATION FORCE

For a constant perturbation force, the steady state solution of equation (13) is of the form of a Duhamel integral [1],

$$q_{dif} = \frac{F_0}{m\beta}\int_0^{t_1} e^{-\alpha(t_1 - t)}\sin[\beta(t_1 - t)]dt, \tag{17}$$

where t_1 is a constant,

$$\beta = \omega\sqrt{1 - D^2} \tag{18}$$

is the circular eigenfrequency of the damped vibration,

$$D = \frac{\alpha}{\omega} \tag{19}$$

is the damping ration and

$$Q(t_1) = F_0. \tag{20}$$

By calculating the integral (17), and by introducing the non-dimensional quantity

$$\zeta = \beta t_1, \tag{21}$$

the following expression is obtained:

$$q_{dif} = \frac{2F_0}{19k}\cdot \\ \cdot\left\{1 - e^{-\frac{D}{\sqrt{1-D^2}}}\left[\frac{D}{\sqrt{1-D^2}}\sin(\zeta) + \cos(\zeta)\right]\right\}. \tag{22}$$

This expression can be rewritten as

$$\hat{q}_{dif} = 1 - e^{-\frac{D}{\sqrt{1-D^2}}}\left[\frac{D}{\sqrt{1-D^2}}\sin(\zeta) + \cos(\zeta)\right], \tag{23}$$

where

$$\hat{q}_{dif} = \frac{19k\cdot q_{dif}}{2F_0}. \tag{24}$$

The function $\hat{q}_{dif}(\zeta)$ is represented in Fig. 2, for various values of the damping ratio D.

The force acting upon the damping system is

$$\mathsf{F} = kq_{dif} + c\dot{q}_{dif}, \tag{25}$$

where, by differentiating (22),

$$\dot{q}_{dif} = \frac{F_0}{m\omega^2}e^{-\alpha\cdot t_1}\left(\frac{\alpha^2}{\beta} + \beta\right)\sin(\beta t_1). \tag{26}$$

By substituting (22) and (26), expression (25) takes the form

$$\mathsf{F} = \frac{2F_0}{19}\left\{1 - e^{-\frac{D}{\sqrt{1-D^2}}}\left[\cos(\zeta) - \frac{D}{\sqrt{1-D^2}}\sin(\zeta)\right]\right\}, \tag{27}$$

or

$$\hat{F}(\zeta,D)=1-e^{-\frac{D}{\sqrt{1-D^2}}}\left[\cos(\zeta)-\frac{D}{\sqrt{1-D^2}}\sin(\zeta)\right], \quad (28)$$

where

$$\hat{F} = F\,\frac{19}{2F_0}. \quad (29)$$

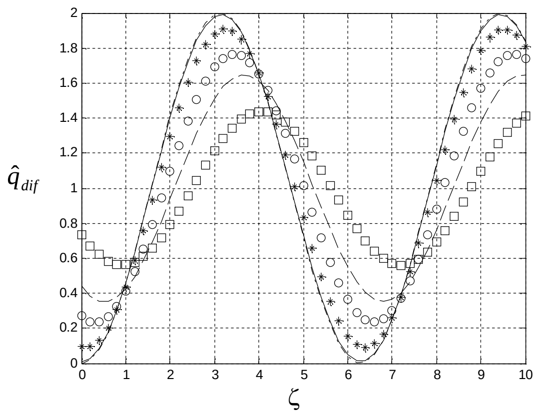

Fig. 2. Variation of function $\hat{q}_{dif}$ for various values of the damping ratio D:
$D=0\ (-\bullet)$, $D=0.01\ (-)$, $D=0.1\ (*)$,
$D=0.3\ (\circ)$, $D=0.5\ (--)$, $D=0.8\ (\square)$

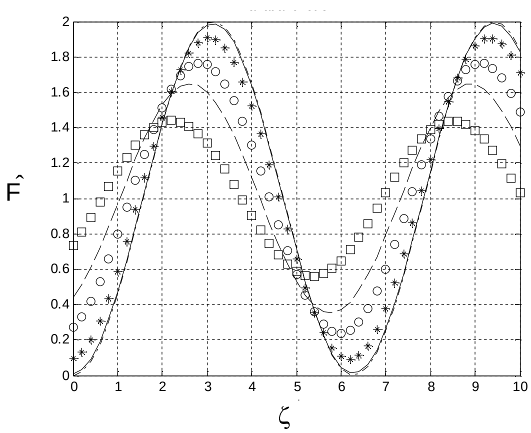

Fig. 3. Variation of function $\hat{F}$ for various values of the damping ratio D:
$D=0\ (-\bullet)$, $D=0.01\ (-)$, $D=0.1\ (*)$,
$D=0.3\ (\circ)$, $D=0.5\ (--)$, $D=0.8\ (\square)$

The force F is represented in Fig. 3 as a function of F_{rigid} .

5. TRIANGULAR PERTURBATION FORCE

For a triangular periodical perturbation force (Fig. 4), which is antisymmetrical ($F(-t)=-F(t)$), the Fourier series expansion has only the sin terms, so that $a_0=0$, $a_k=0$ and for $T=\frac{2\pi}{\Omega}$ [4]:

$$b_k=\frac{2\Omega}{\pi}\int_0^{\frac{\pi}{\Omega}}F(t)\cdot\sin(k\Omega t)\,\mathrm{d}t=\frac{8A_0}{k^2\pi^2}\sin\left(k\frac{\pi}{2}\right).$$

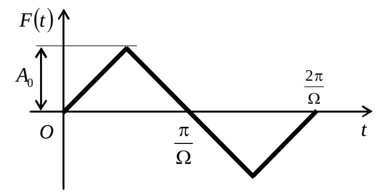

Fig. 4. Triangular perturbation force

The Fourier series becomes:

$$F(t)=\frac{8A_0}{\pi^2}\cdot\left[\sin(\Omega t)-\frac{1}{9}\sin(3\Omega t)+\frac{1}{25}\sin(5\Omega t)-...\right], \quad (30)$$

that allows to write the differential equation of motion (13) in the form:

$$\ddot{q}_{dif}+2\alpha\dot{q}_{dif}+\omega^2 q_{dif}=\frac{8A_0}{m\pi^2}\left[\sin(\Omega t)-\frac{1}{9}\sin(3\Omega t)+\frac{1}{25}\sin(5\Omega t)-...\right]. \quad (31)$$

For the undamped oscillation $(\alpha=0)$,

$$q_{dif}=\sum_{k=0}^{n}A_{2k+1}\sin[(2k+1)\Omega t], \quad (32)$$

which, introduced in the differential equation, allows to determine the coefficients

$$A_{2k+1}=\frac{8A_0}{\pi^2 n}\cdot\frac{1}{(2k+1)^2\left[1-\frac{(2k+1)^2\Omega^2}{\omega^2}\right]} \quad (33)$$

$$(k=0,1,2,...,n),$$

i.e. the expressions A_1, A_2, A_3,…, for various values of the ratio $\frac{\Omega}{\omega}$. From the analysis of the obtained results, it follows that, for certain values of this ratio, the coefficients of the series (32) decrease sharply (for example, $\frac{A_3}{A_1}=0.00297$ if $\frac{\Omega}{\omega}=0.95$). Consequently, the study of the damped system can be made in a first order approximation, on the equation

$$\ddot{q}_{dif} + 2\alpha\dot{q}_{dif} + \omega^2 q_{dif} = \frac{8A_0}{m\pi^2}\sin(\Omega t). \quad (34)$$

The solution of the homogeneous linear differential equation is

$$q_{h\,\mathrm{dif}} = e^{-\alpha \cdot t}\left[C_1\cos(\beta t) + C_2\sin(\beta t)\right], \quad (35)$$

where C_1, C_2 are integration constants, while the particular solution of the non-homogeneous equation is

$$q_{p\,\mathrm{dif}} = A\sin(\Omega t - \varphi), \quad (36)$$

with

$$\begin{cases} A = \dfrac{8A_0}{m\pi^2\omega^2} \cdot \dfrac{1}{\sqrt{\left(1-\eta^2\right)^2 + 4D^2\eta^2}} \\ \tan\varphi = \dfrac{2\alpha\Omega}{\omega^2 - \Omega^2}, \end{cases} \quad (37)$$

where

$$\eta = \frac{\Omega}{\omega} \quad (38)$$

The general solution of equation (34) is

$$q_{\mathrm{dif}} = e^{-\alpha \cdot t}\left[C_1\cos(\beta t) + C_2\sin(\beta t)\right] + A\sin(\Omega t - \varphi). \quad (39)$$

Considering

$$\eta = 0.95, \quad (40)$$

the integration constants were determined from the initial conditions (16), resulting

$$q_{\mathrm{dif}} = 0.0853\frac{A_0}{k}\frac{1}{\sqrt{0.0095 + 3.61D^2}} \cdot \sin\left[\frac{0.95}{\sqrt{1-D^2}}\zeta - \arctan(19.4871D)\right],$$

$$\dot{q}_{\mathrm{dif}} = 0.249\frac{A_0}{\sqrt{km}}\frac{1}{\sqrt{0.0095 + 3.61D^2}} \cdot \cos\left[\frac{0.95}{\sqrt{1-D^2}}\zeta - \arctan(19.4871D)\right], \quad (41)$$

The function $\hat{q}_{dif}(\zeta,)$, where

$$\hat{q}_{dif} = \frac{k \cdot q_{\mathrm{dif}}}{0.0853A_0}, \quad (42)$$

is represented in Figure 5, for various values of the damping ratio D.

A procedure similar to the one used for the non-dimensionlization (29) leads, in this case, to

$$\hat{F} = 11.723F/A_0. \quad (43)$$

The function $\hat{F}(\zeta)$ is represented in Figure 6 for various values of the damping ratio D.

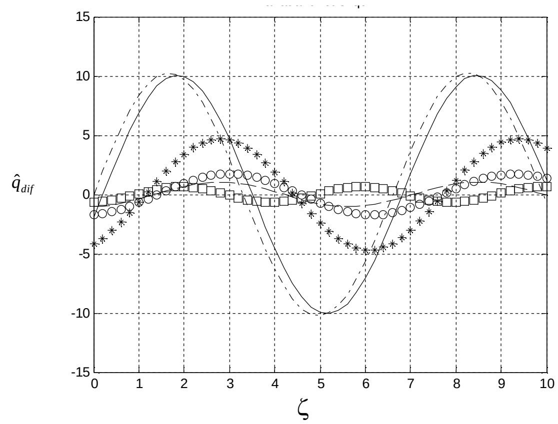

Fig. 5. Variation of function $\hat{q}_{dif}$ for various values of the damping ratio D: $D = 0\ (-\bullet)$, $D = 0.01\ (-)$, $D = 0.1\ (*)$, $D = 0.3\ (\circ)$, $D = 0.5\ (--)$, $D = 0.8\ (\square)$

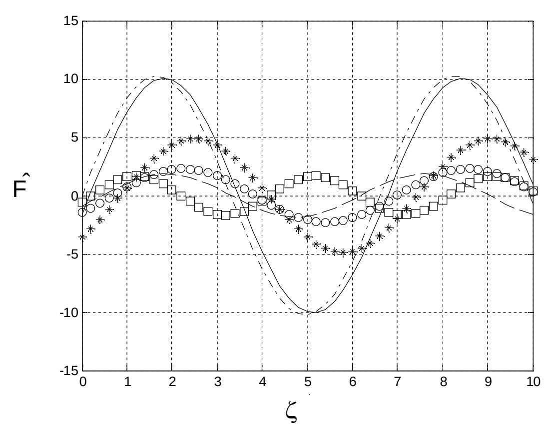

Fig. 6. Variation of function $\hat{F}$ for various values of the damping ratio D: $D = 0\ (-\bullet)$, $D = 0.01\ (-)$, $D = 0.1\ (*)$, $D = 0.3\ (\circ)$, $D = 0.5\ (--)$, $D = 0.8\ (\square)$

6. RESULTS AND CONCLUSIONS

It can be seen from the numerical studies presented above, that the greatest relative displacement and the greatest value of the non-dimensional force are obtained for $D = 0$

and both $\left(\hat{q}_{dif}\right)_{\max}$ and $\left(\hat{\mathsf{F}}\right)_{\max}$ decrease with the damping ratio.

For the constant perturbation force, these values are

$$\left(\hat{q}_{dif}\right)_{\max} = 2 \,, \tag{44}$$

$$\left(\hat{\mathsf{F}}\right)_{\max} = 2 \tag{45}$$

and

$$\left(q_{dif}\right)_{\max} = \frac{4F_0}{19k}, \tag{46}$$

$$\left(\mathsf{F}\right)_{\max} = \frac{4F_0}{19}, \tag{47}$$

while the highest value of the force acting upon the damping system occurs at approximately $\zeta = 3$, which corresponds to the time $t_1 = \frac{\zeta}{\beta}$.

For the triangular perturbation force,

$$\left(\hat{q}_{dif}\right)_{\max} = 10 \,, \tag{48}$$

$$\left(\hat{\mathsf{F}}\right)_{\max} = 10 \tag{49}$$

and

$$\left(q_{dif}\right)_{\max} = 0.853\frac{A_0}{k}, \tag{50}$$

$$\left(\mathsf{F}\right)_{\max} = 0.853A_0 \,, \tag{51}$$

while the highest value of the force acting upon the damping system occurs at approximately $\zeta = 2$, which also corresponds to the time $t_1 = \frac{\zeta}{\beta}$.

The analytical model and the numerical study presented in the paper are useful for the calibration and optimization of the medical apparatus, in order to satisfy optimally meet the necessities of the therapeutical process.

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