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THE APPROXIMATE OR EXACT ANALYTICAL SOLUTIONS FOR THE DIFFERENTIAL EQUATIONS WHICH DESCRIBE PARAMETRIC FREE VIBRATIONS OF MECHANICAL SYSTEMS

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ABSTRACT

The vibrations of many mechanical systems can be described by differential equations that have coefficients that depend on time. It is difficult to determine the approximate or exact analitical solutions of these equations.
In this work, we show that we can obtain analytical exact or approximate solutions for non-linear homogeneous differential equations with variable coefficient which describes the parametric free vibrations of mechanical systems.

KEYWORDS: mechanical system, non-linear vibrations

1. INTRODUCTION

There are mechanical systems that have variable parameters in time such as the mass, the dimensions, the elastic coefficient or the damping coefficient. The oscillations of a system with variable parameters are described by homogeneous differential equation

$$\ddot{x} + f(\dot{x}, x, t) = 0 \tag{1}$$

In [1] we have shown that we can find analytical approximate solutions for non-linear differential equations as

$$\ddot{x} + f(x,t) = 0 \tag{2}$$

with

$$f(x,t) = -c \cdot t^k \cdot x;\ c,k \in R \tag{3}$$

which describe the parametrical free vibrations of the mechanical systems that do not imply damping forces.

In [2] we analysed the parametric vibrations of the elastic systems, using the answer in time when the vibrations are described by a homogeneus non-linear differential equation

$$\begin{aligned} &\ddot{x} + \delta x^2 \cdot \dot{x} + \alpha\, sgn\, \dot{x} + \\ &+ \left[1 + \mu \cos 2(\omega t - \phi)\right] x + \beta x^3 = 0 \end{aligned} \tag{4}$$

Non-autonomous vibrations described by non-linear differential equation

$$\ddot{x} + f(\dot{x}, t) = 0\,. \tag{5}$$

In [3] we demonstrated that for functions such as

$$f\left(\dot{x}, t\right) = a\,\dot{x}^2 + \frac{1}{t^2}\left(bt^{\alpha} + c\right);\quad a,b,c,\alpha \in R \tag{6}$$

we can determine the analytical exact solutions for non-linear differential equations (5) .

We now show that we can find for the differential equation (2) analitical exact or aproximate solutions, using the Bessel functions.

2. THEORETICAL CONCEPT

After substituting f(x,t) from (3), the differential equation (2) becomes

$$\ddot{x} - c \cdot t^k \cdot x = 0 \qquad (7)$$

and it is a particular case of the differential equation

$$t^2 \cdot \frac{d^2u}{dt^2} + a \cdot t \cdot \frac{du}{dt} + \left(b + c \cdot t^m\right) \cdot u = 0 \cdot \qquad (8)$$

We will show that it comes from a Bessel differential equation

$$x^2 \cdot y'' + x \cdot y' + \left(x^2 - p^2\right) \cdot y = 0 \cdot \qquad (9)$$

Adding a new independent variable *t* and a new function *u* [2]

$$y = t^\alpha \cdot u \ ; x = \gamma \cdot t^\beta \qquad (10)$$

with $\alpha, \beta, \gamma \in R$, which are not null, we obtain

$$\frac{dt}{dx} = \frac{1}{\beta\gamma} t^{1-\beta} \ ;$$

$$\frac{dy}{dx} = \frac{1}{\beta\gamma} t^{1-\beta} \cdot \frac{dy}{dt} \ ;$$

$$\frac{d^2y}{dx^2} = \frac{1}{\beta\gamma} t^{1-\beta} \left(\frac{1}{\beta\gamma} t^{1-\beta} \frac{d^2y}{dt^2} + \frac{1-\beta}{\beta\gamma} \cdot t^{-\beta} \cdot \frac{dy}{dt} \right) \ ;$$

$$\frac{dy}{dt} = t^\alpha \cdot \frac{du}{dt} + \alpha \cdot t^{\alpha-1} \cdot u \ ; \qquad (11)$$

$$\frac{d^2y}{dt^2} = t^\alpha \cdot \frac{d^2u}{dt^2} + 2\alpha \cdot t^{\alpha-1} \cdot \frac{du}{dt} + \alpha\left(\alpha - 1\right) \cdot t^{\alpha-2} \cdot u$$

Replacing y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in the differential equation (9) and $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ in $u, \frac{du}{dt}, \frac{d^2u}{dt^2}$ than

$$t^2 \cdot \frac{d^2u}{dt^2} + \left(2\alpha + 1\right) \cdot t \cdot \frac{du}{dt} + \left(\alpha^2 - \beta^2 p^2 + \beta^2\gamma^2 \cdot t^{2\beta}\right) \cdot u = 0. \qquad (12)$$

If *p* is not a whole number and it is not half of a whole number [4], than the general integer of the differential equation (9) is

$$y = c_1 \cdot J_p\left(x\right) + c_2 \cdot J_{-p}\left(x\right) \ ; \quad c_1, c_2 \in R \qquad (13)$$

where

$$J_p\left(x\right) = \sum_{s=0}^{\infty} \frac{\left(-1\right)^s}{s!\,\Gamma\left(p+s+1\right)} \cdot \left(\frac{x}{2}\right)^{p+2s} \qquad (14)$$

is the Bessel function and Γ is the Euler function.

If p=n is a whole pozitive number, then

$$J_n\left(x\right) = \sum_{s=0}^{\infty} \frac{\left(-1\right)^s}{s!\left(n+s\right)!} \cdot \left(\frac{x}{2}\right)^{n+2s} \qquad (15)$$

is a solution for the differential equation (9) with

$$K_n\left(x\right) = \beta \cdot J_n\left(x\right) \cdot \ln x + x^{-n} \sum_{s=0}^{\infty} \beta_s x^s \qquad (16)$$

where $\beta, \beta_0, \beta_1, \ldots \in R$.

The general integer of the equation (2) will be

$$y = c_1 \cdot J_n\left(x\right) + c_2 K_n\left(x\right) \ ; \quad c_1, c_2 \in R \cdot \qquad (17)$$

If *p* is half of a whole odd number,

$$p = \frac{2n+1}{2}, \qquad (17')$$

the general integer of the equation (2) is like (3), where

$$J_{\frac{2n+1}{2}}\left(x\right) = \sqrt{\frac{2}{\pi x}} \left[P_n\left(\frac{1}{x}\right) \sin x + Q_n\left(\frac{1}{x}\right) \cos x \right] \qquad (18)$$

with $P_n\left(\frac{1}{x}\right)$, $Q_n\left(\frac{1}{x}\right)$ polynomial in $\frac{1}{x}$.

Because the differential equation (9) has the general integer (13), the differential equation (12) will have

$$u = t^{-\alpha} \left[c_1 J_p\left(\gamma \cdot t^\beta\right) + c_2 J_{-p}\left(\gamma \cdot t^\beta\right) \right] \qquad (19)$$

in which we replace $J_{-p}\left(\gamma \cdot t^\beta\right)$ with $K_n\left(\gamma \cdot t^\beta\right)$ if p=n is zero or a whole and pozitive number.

The differential equation (12) is like (8) for

$$(2\alpha+1)=a\,,\ \alpha^2-\beta^2p^2=b\,,$$

$$\beta^2\gamma^2=c\,,\ 2\beta=m\,. \qquad (20)$$

For any differential equation like (9) where the invariable c and m are not null we can find using (20) the values of α,β,γ,p and the general integer of the differential equation (12) will be expressed using the Bessel functions after formulas (13) and (17).

3. EXAMPLES

We exemplify now for elastic systems whose parametric oscillations are characterized by differential equations (2) that verify the condition (3).

1. The parametric free vibrations of a mechanical system are described by the differential equation

$$\ddot{x}+\frac{2}{9}\cdot x\cdot t^{-4}=0\,, \qquad (21)$$

an eqution like (8) with

$$a=b=0;c=\frac{2}{9}\,, \qquad (22)$$

and from (20) we obtain

$$\alpha=-\frac{1}{2}\,,p=\frac{1}{2}\,,\beta=-1\,,\gamma=\frac{\sqrt{2}}{3}\,,m=-2\,. \qquad (23)$$

The invariable p is half of a whole odd number, than $n=0$ and differential equation (21) has the general analytical exact solution

$$x(t)=t^{\frac{1}{2}}\left[c_1\cdot J_{\frac{1}{2}}\left(\frac{\sqrt{2}}{3t}\right)+c_2\cdot J_{-\frac{1}{2}}\left(\frac{\sqrt{2}}{3t}\right)\right]= \qquad (24)$$

$$=t\sqrt{\frac{6}{\pi\sqrt{2}}}\left(c_1\cdot\sin\frac{\sqrt{2}}{3t}+c_2\cdot\cos\frac{\sqrt{2}}{3t}\right),$$

$$c_1,c_2\in R\,,$$

with

$$J_{\frac{1}{2}}(\mathrm{v})=\sqrt{\frac{2}{\pi\mathrm{v}}}\sin\mathrm{v}\,,$$

$$J_{-\frac{1}{2}}(\mathrm{v})=\sqrt{\frac{2}{\pi\mathrm{v}}}\cos\mathrm{v}\,. \qquad (25)$$

2. The parametric free oscillations of a mechanical system have as mathematical model the non–linear differential equation

$$\ddot{x}-\frac{1}{4}\cdot x\cdot t^{-5}=0 \qquad (26)$$

an eqution like (8) with

$$a=b=0;c=\frac{1}{4}\,, \qquad (27)$$

and from (20) we obtain

$$\alpha=-\frac{1}{2}\,,p=\frac{3}{2}\,,\beta=-\frac{2}{3}\,,\gamma=\frac{3}{4}\,,m=\frac{4}{3}\,. \qquad (28)$$

The invariable p is half of a whole odd number, than $n=1$ and differential equation (21) has the general analytical exact solution

$$x(t)=t^{\frac{1}{2}}\left[c_1\cdot J_{\frac{3}{2}}\left(\frac{3}{4}t^{\frac{2}{3}}\right)+c_2\cdot J_{-\frac{3}{2}}\left(\frac{3}{4}t^{\frac{2}{3}}\right)\right]=$$

$$=2\sqrt{\frac{2}{3\pi}}t^{\frac{1}{6}}\left[c_1\left(\frac{4}{3}t^{-\frac{2}{3}}\sin\frac{3}{4}t^{\frac{2}{3}}-\cos\frac{3}{4}t^{\frac{2}{3}}\right)\right]-$$

$$-2\sqrt{\frac{2}{3\pi}}t^{\frac{1}{6}}\left[c_2\left(\sin\frac{3}{4}t^{\frac{2}{3}}+\frac{4}{3}t^{-\frac{2}{3}}\cos\frac{3}{4}t^{\frac{2}{3}}\right)\right], \qquad (29)$$

with

$$J_{\frac{3}{2}}(\mathrm{v})=\sqrt{\frac{2}{\pi\mathrm{v}}}\left(\frac{1}{v}\sin v-\cos v\right)\,,$$

$$J_{-\frac{3}{2}}(\mathrm{v})=-\sqrt{\frac{2}{\pi\mathrm{v}}}\left(\sin v+\frac{1}{v}\cos\mathrm{v}\right) \qquad (30)$$

and $c_1,c_2=const$.

The invariable c_1, c_2 that appear in (24), (29) are in the initial condition that was imposed to the system’s movement.

Observations : a. In the solved differential equations we have obtained exact analytical solutions in the examples shown, but generally we will find analytical approximate solutions using Bessel functions.

b. We shall now take the special Riccati differential equation [4] of the first order

$$y'+k_1\cdot y^2-k_2\cdot x^k=0 \qquad (31)$$

with $k, k_1, k_2 \in R$, about which we know that we can only find solutions in some special situations.

Using the substitution

$$k_1 \cdot y = \frac{1}{u} \cdot \frac{du}{dx} \qquad (32)$$

the differential equation (31) becomes

$$x^2 \cdot \frac{d^2u}{dx^2} - k_1 k_2 x^{k+2} \cdot u = 0 \qquad (33)$$

because with the derivation of the substitution (31) we obtain

$$k_1 \cdot y' = -\frac{1}{u^2} \cdot \left(\frac{du}{dx}\right)^2 + \frac{1}{u} \cdot \frac{d^2u}{dx^2} \cdot \qquad (34)$$

The equation (33) is like (8) where

$$a=b=0,\ c=-k_1 \cdot k_2,\ m=k+2 \qquad (35)$$

and its solutions are Bessel functions if the invariables k_1, k_2 have opposite signs for $c \geq 0$, as (20).

After solving the function *u* from the equation (33) with (32) we can find the general soluton *y* of the equation (31).

The differential equation from the first example

$$\ddot{x} + \frac{2}{9} \cdot x \cdot t^{-4} = 0 \qquad (36)$$

which is equal to the special Riccati differential equation

$$y' = \frac{1}{3} \cdot y^2 + \frac{2}{3} \cdot x^{-4} \qquad (37)$$

such as (31) where

$$k_1 = -\frac{1}{3}, k_2 = \frac{2}{3}, k = -4 \qquad (38)$$

Derivating $x(t)$ from (24) and using the substitution (32) we obtain the general exact analytical solution of the Riccati differential equation (37)

$$y(x) = -3 \cdot x^{-1} + \sqrt{2} \cdot x^{-2} \cdot \frac{c_0 \cdot \cos\frac{\sqrt{2}}{3x} - \sin\frac{\sqrt{2}}{3x}}{c_0 \cdot \sin\frac{\sqrt{2}}{3x} + \cos\frac{\sqrt{2}}{3x}} =$$

$$= -\frac{3}{x} + \frac{\sqrt{2}}{x^2} \cdot \frac{c_0 - tg\frac{\sqrt{2}}{3x}}{c_0 tg\frac{\sqrt{2}}{3x} + 1}, \qquad (39)$$

where $c_0 = \frac{c_1}{c_2} = const.$

5. CONCLUSIONS

For the non-linear differential equations of second order that describe the free parametric oscillations of mechanical systems there are situations when we can obtain analytical exact or approximate solutions, using Bessel functions that offer the possibility to study the movement of the respective systems.

The Bessel functions are useful when obtaining the analytical approximate or exact solutions for the special Riccati differential equations of the first order.

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