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ASPECTS REGARDING THE STUDY OF A VIBRATORY DEVICE WHICH ACTS A MEDICAL APPARATUS

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ABSTRACT

The paper studies, from the electromechanical point of view, a medical apparatus designed for spine treatment by vibrations. The differential equations of motion of the system have been obtained by using Lagrange equations of the second species, in the form specific to electromechanical systems with electrical part coupled to the mechanical one. The obtained nonlinear differential system has been integrated numerically, considering the cases of undamped and damped free vibrations, respectively.

KEYWORDS: medical apparatus, electromechanical device, biomechanics, mechanical vibrations

1. GENERAL CONSIDERATIONS

The paper studies a medical apparatus designed for spine treatment by vibrations produced by an electromechanical device (Fig. 1).

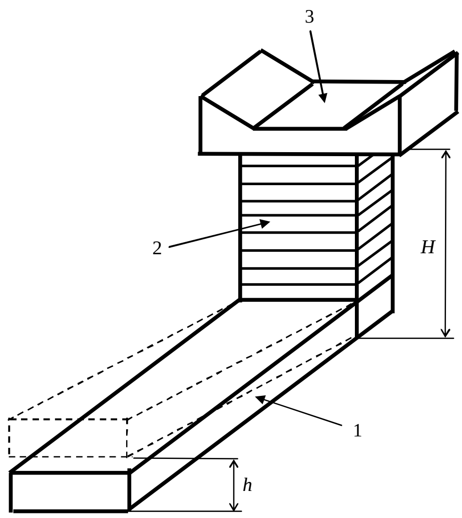

Fig. 1. Medical apparatus for treatment by vibrations

The main parts of the system are:

1. adjustable platform, with various inclination angles with respect to the horizontal plane, for supporting the body;
2. elastic column, that contains the electromechanical device, which produces and transmits the vibrations;
3. leg support.

Depending on the height of the subject, the elastic column 2 can be fixed at a certain distance H from the horizontal plane.

The device transmits to the spine vibrations of a certain frequency, that contribute to its fortification, diminishes the risk of beak growth and they stop the evolution of some diseases [2].

Taking into account the purpose of the paper, the analysis of the electromechanical device must consider certain frequency domains, so that it fulfils medical prescriptions and it does not harm the organ upon which it acts, by nonadequate values that could produce resonance phenomena.

2. STUDY MODEL

For the study, the model of an electromechanical device is considered (Fig. 2), consisting of the plate 4, in contact with the support 3. The plate is connected to the fixed part of the device by means of the elastic elements with the constants k and k' and of the system of four hinged bars, each with mass M and length l. The hinges are considered material points with mass m. The plate is acted by the movable plunger 5 of a coil 7, with the fixed magnetic core 6. The coil is connected to constant voltage source u and to a linear resistor R. The motion of the movable plunger modifies the height s of the plane air gap and, consequently, the magnetic inductance, which produces the variation of the current in the circuit. This variation can be observed on the oscilloscope 8.

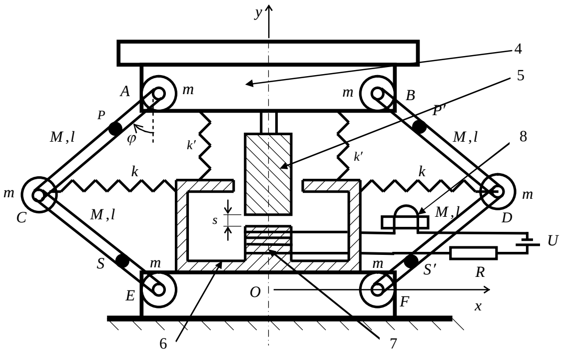

Fig. 2. Mechanical model

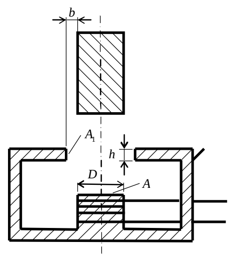

Fig. 3. Significant dimensions

The variation of the inductance is [3]

$$L = \frac{\mu_0 N^2}{\dfrac{2b}{A_1} + \dfrac{s}{A_0}} = \frac{\alpha}{1+\beta x} = \frac{\alpha}{1+\beta(2l\cos\varphi - h_0)}, \quad (1)$$

where the following notations were used (Fig. 3):

$\mu_0 = 4\pi \times 10^{-7}\ \dfrac{\text{Vs}}{\text{Am}}$ – magnetic constant;

N – number of turns;

d – loop diameter;

h – height of the cylindrical air gap;

b – width of the cylindrical air gap;

$A_0 = \dfrac{\pi d^2}{4}$ – area of the plane air gap;

$A_1 = \pi d h$ – area of the cylindrical air gap;

h_0 – total height of the movable plunger and of the central part of the magnetic core.

The constants α and β in formula (1) have the expressions:

$$\alpha = \frac{\mu_0 N^2 A_1}{2b}, \quad (2)$$

$$\beta = \frac{1}{2b}\frac{A_1}{A_0}. \quad (3)$$

Plate 4 has the mass M_1, which contains also the part it supports, of the mass of the subject.

For a constant voltage u, a static equilibrium position is obtained, influenced by the attraction force

$$F_0 = \frac{\partial}{\partial s}\left(\frac{L i_0^2}{2}\right), \quad (4)$$

where

$$i_0 = \frac{u}{R}. \quad (5)$$

3. FREE UNDAMPED VIBRATIONS

The model of the electromechanical system shown in Figure 2 has two degrees of freedom, defined by the following independent parameters: angle φ and electric charge q.

The differential equations of the free undamped vibrations of the system with electrical part coupled to the mechanical one [3], [4], [6] can be obtained by using Lagrange equations of second species,

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial E}{\partial \dot\varphi}\right) - \dfrac{\partial E}{\partial \varphi} = Q_\varphi \\ \dfrac{d}{dt}\left(\dfrac{\partial E}{\partial \dot q}\right) - \dfrac{\partial E}{\partial q} = Q_q, \end{cases} \quad (6)$$

where E is the kinetic, for a while Q_φ and Q_q are the generalized forces.

The generalized forces can be expressed by using the force function U and the dissipation function Φ:

$$\begin{cases} Q_\varphi = \dfrac{\partial U}{\partial \varphi} + \dfrac{\partial \Phi}{\partial \dot\varphi} \\ Q_q = \dfrac{\partial U}{\partial q} + \dfrac{\partial \Phi}{\partial \dot q}. \end{cases} \quad (7)$$

In order to obtain the differential equations, first the coordinates of the significant points of system are written:

- the center of mass C_1 of the plate 4,

$$\begin{cases} x_{C_1} = 0 \\ y_{C_1} = 2l\cos\varphi + h_1, \end{cases} \tag{8}$$

where h_1 is the height of point C_1 with respect to hinges A and B;

- a point S on the bar EC, situated at the distance λ from the hinge E,

$$\begin{cases} x_S = -\dfrac{l_1}{2} - \lambda\sin\varphi \\ y_S = \lambda\cos\varphi, \end{cases} \tag{9}$$

where $l_1 = EF = AB$;

- a point S' on the bar FD, situated at the distance λ from the hinge E,

$$\begin{cases} x_{S'} = \dfrac{l_1}{2} + \lambda\sin\varphi \\ y_{S'} = \lambda\cos\varphi; \end{cases} \tag{10}$$

- a point P on the bar AC, situated at the distance λ from the hinge A,

$$\begin{cases} x_P = -\dfrac{l_1}{2} - \lambda\sin\varphi \\ y_P = (2l - \lambda)\cos\varphi; \end{cases} \tag{11}$$

- a point P' on the bar BD, situated at the distance λ from the hinge B,

$$\begin{cases} x_{P'} = \dfrac{l_1}{2} + \lambda\sin\varphi \\ y_{P'} = (2l - \lambda)\cos\varphi. \end{cases} \tag{12}$$

By differentiation, the components of the velocities can be obtained and, subsequently, their moduli:

$$v_{C_1}^2 = \dot{x}_{C_1}^2 + \dot{y}_{C_1}^2 = 4l^2\dot{\varphi}^2\sin^2\varphi, \tag{13}$$

$$v_S^2 = v_{S'}^2 = \dot{x}_{S'}^2 + \dot{y}_{S'}^2 = \lambda^2\dot{\varphi}^2, \tag{14}$$

$$v_P^2 = v_{P'}^2 = \dot{x}_{P'}^2 + \dot{y}_{P'}^2 = \left[\lambda^2 + 4l(l-\lambda)\sin^2\varphi\right]\dot{\varphi}^2. \tag{15}$$

The velocities of the points C and D can be obtained from formula (14), choosing $\lambda = l$:

$$v_C^2 = v_D^2 = l^2\dot{\varphi}^2. \tag{16}$$

Similarly, the velocity of the mass center C_2 of the bar AC can be obtained from formula (15), choosing $\lambda = \dfrac{l}{2}$:

$$v_{C_2}^2 = \left(\frac{1}{4} + 2\sin^2\varphi\right) l^2\dot{\varphi}^2. \tag{17}$$

These velocities are used to determine the kinetic energy of the elements of the system, as follows:

- for the plate,

$$E_1 = \frac{1}{2}(M_1 + 2m){v_{C_1}}^2 = 2(M_1 + 2m)l^2\dot{\varphi}^2; \tag{18}$$

- for bars EC and FD,

$$E_{EC} = E_{FD} = \frac{1}{2}\frac{Ml^2}{3}\dot{\varphi}^2 = \frac{Ml^2}{6}\dot{\varphi}^2; \tag{19}$$

- for bars AC and BD,

$$\begin{aligned} E_{AC} = E_{BD} &= \frac{1}{2}Mv_{C_2}^2 + \frac{1}{2}\frac{Ml^2}{3}\dot{\varphi}^2 = \\ &= \frac{Ml^2}{24}\left(4 + 9\sin^2\varphi\right)\dot{\varphi}^2; \end{aligned} \tag{20}$$

- for the hinges C and D,

$$E_C = E_D = \frac{1}{2}mv_C^2 = \frac{ml^2}{2}\dot{\varphi}^2. \tag{21}$$

Taking into account also the energy of the magnetic field $E_L = \dfrac{1}{2}L\cdot\dot{q}^2$ [3], [4], the kinetic energy of the whole system is

$$\begin{aligned} E &= E_1 + 2E_{EC} + 2E_{AC} + 2E_C + E_L = \\ &= \frac{1}{2}\left(A\sin^2\varphi + B\right)\dot{\varphi}^2 + \frac{1}{2}L(\varphi)\dot{q}^2, \end{aligned} \tag{22}$$

where

$$\begin{cases} A = 4(M_1 + 2m + M)l^2 \\ B = 2\left(\dfrac{2}{3}M + m\right)l^2. \end{cases} \tag{23}$$

The force function corresponding to the weights, to the elastic forces, and to the voltage source, is

$$\begin{aligned} U = &-D\cos\varphi - k(l\sin\varphi - x_0)^2 - \\ &-k'(l\cos\varphi - y_0)^2 + qu + C, \end{aligned} \tag{24}$$

where x_0 and y_0 are the undeformed lengths of the horizontal and vertical springs, respectively, while

$$D = 2(2M + 3m + M_1)gl. \tag{25}$$

The dissipation function due to the resistor is:

$$\Phi = -\frac{1}{2} R\dot{q}^2 . \quad (26)$$

Calculating and replacing the derivatives, equations (6) take the form

$$\begin{cases} \left(A\sin^2\varphi + B\right)\ddot{\varphi} + A\dot{\varphi}^2 \sin\varphi\cos\varphi - \frac{1}{2}\frac{\partial L}{\partial\varphi}\dot{q}^2 - \\ \quad - D\sin\varphi + 2k\left(l\sin\varphi - x_0\right)l\cos\varphi - \\ \quad - 2k'\left(l\cos\varphi - y_0\right)l\sin\varphi = 0 \\ L(\varphi)\ddot{q} + \frac{\partial L}{\partial\varphi}\dot{\varphi}\dot{q} + R\dot{q} = u. \end{cases} \quad (27)$$

By replacing the derivative of the inductance,

$$\frac{\partial L}{\partial\varphi} = \frac{2\alpha\beta l\sin\varphi}{\left[1+\beta(2l\cos\varphi - h_0)\right]^2}, \quad (28)$$

system (27) becomes

$$\begin{cases} \left(A\sin^2\varphi + B\right)\ddot{\varphi} + A\dot{\varphi}^2 \sin\varphi\cos\varphi - \\ \quad - \frac{\alpha\beta l\sin\varphi}{\left[1+\beta(2l\cos\varphi - h_0)\right]^2}\dot{q}^2 - \\ \quad - D\sin\varphi + 2k\left(l\sin\varphi - x_0\right)l\cos\varphi - \\ \quad - 2k'\left(l\cos\varphi - y_0\right)l\sin\varphi = 0 \\ \frac{\alpha}{1+\beta(2l\cos\varphi - h_0)}\ddot{q} + \\ \quad + \frac{2\alpha\beta l\sin\varphi}{\left[1+\beta(2l\cos\varphi - h_0)\right]^2}\dot{\varphi}\dot{q} + R\dot{q} = u. \end{cases} \quad (29)$$

This system of two second-order differential equations is equivalent to another one, consisting of three first-order differential equations:

$$\begin{cases} \dot{\varphi} = \omega \\ \dot{\omega} = \frac{1}{A\sin^2\varphi + B}\left\{-A\omega^2\sin\varphi\cos\varphi + \right. \\ \quad + \frac{\alpha\beta l\sin\varphi}{\left[1+\beta(2l\cos\varphi - h_0)\right]^2} i^2 + \\ \quad + D\sin\varphi - 2k\left(l\sin\varphi - x_0\right)l\cos\varphi + \\ \quad \left. + 2k'\left(l\cos\varphi - y_0\right)l\sin\varphi\right\} \\ \dot{i} = \frac{1+\beta(2l\cos\varphi - h_0)}{\alpha} \times \\ \quad \times\left\{u - \frac{2\alpha\beta l\sin\varphi}{\left[1+\beta(2l\cos\varphi - h_0)\right]^2}\omega i - Ri\right\}. \end{cases} \quad (30)$$

This system can be integrated numerically [1].

4. EQUILIBRIUM POSITION

The static equilibrium position of the system [5], necessary for dimensioning the length h_0, can be determined from (30), by replacing:

$$\begin{cases} \varphi = \varphi_0 = \text{const} \\ \omega = 0, \\ i = i_0 = \text{const}. \end{cases} \quad (31)$$

It follows successively:

$$\begin{cases} -\frac{1}{2}\left.\frac{\partial L}{\partial\varphi}\right|_{\varphi=\varphi_0} i_0^2 - D\sin\varphi_0 + \\ \quad + 2k\left(l\sin\varphi_0 - x_0\right)l\cos\varphi_0 - \\ \quad - 2k'\left(l\cos\varphi_0 - y_0\right)l\sin\varphi_0 = 0 \\ Ri_0 = u. \end{cases} \quad (32)$$

$$\begin{aligned} &\frac{\alpha\beta l\sin\varphi_0}{\left[1+\beta(2l\cos\varphi_0 - h_0)\right]^2} i_0^2 - D\sin\varphi_0 + \\ &\quad + 2k\left(l\sin\varphi_0 - x_0\right)l\cos\varphi_0 - \\ &\quad - 2k'\left(l\cos\varphi_0 - y_0\right)l\sin\varphi_0 = 0. \end{aligned} \quad (33)$$

The transcendental equation (33) can be solved numerically.

5. FREE DAMPED VIBRATIONS

The viscous damping of the system in Fig. 2, achieved by connecting bars EC and FD to the fixed element by viscous dampers, manifests itself by the presence of a term $c\dot{\varphi}$ in the first equation (27):

$$\begin{cases} \left(A\sin^2\varphi + B\right)\ddot{\varphi} + A\dot{\varphi}^2 \sin\varphi\cos\varphi - \frac{1}{2}\frac{\partial L}{\partial\varphi}\dot{q}^2 + \\ \quad + c\dot{\varphi} - D\sin\varphi + 2k\left(l\sin\varphi - x_0\right)l\cos\varphi - \\ \quad - 2k'\left(l\cos\varphi - y_0\right)l\sin\varphi = 0 \\ L(\varphi)\ddot{q} + \frac{\partial L}{\partial\varphi}\dot{\varphi}\dot{q} + R\dot{q} = u. \end{cases} \quad (34)$$

This system can be brought to a form similar to (30) that can be integrated numerically.

6. NUMERICAL APPLICATION

The following numerical values have been considered:

$$M_1 = 50\,\text{kg},\ m = 1\,\text{kg},\ M = 2\,\text{kg},$$

$$l = 0.5\,\text{m},\ x_0 = 0.35\,\text{m},\ y_0 = 0.35\,\text{m},\ h_0 = 0.66\,\text{m},$$

$$k = 25000\frac{\mathrm{N}}{\mathrm{m}},\ k' = 25000\frac{\mathrm{N}}{\mathrm{m}},$$

$$u = 220V,\ R = 220\Omega,\ h = 0.02\,\mathrm{m},\ b = 0.002\,\mathrm{m},$$

$$\mu_0 = 4\pi\times10^{-7}\frac{\mathrm{Vs}}{\mathrm{Am}},\ N = 200,\ d = 0.1\,\mathrm{m}.$$

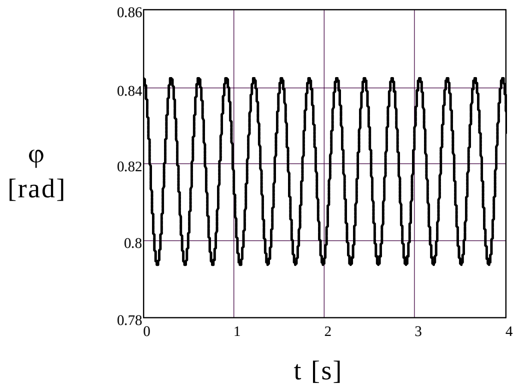

Fig. 4. Free undamped vibration - variation of φ

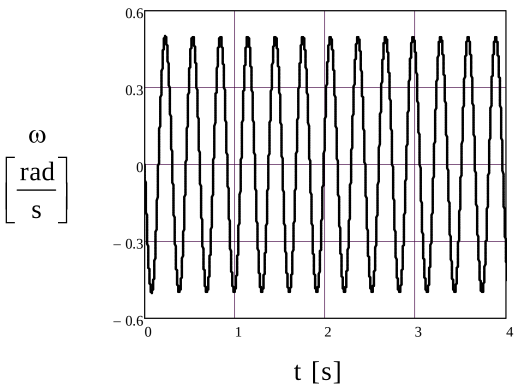

Fig. 5. Free undamped vibration - variation of ω

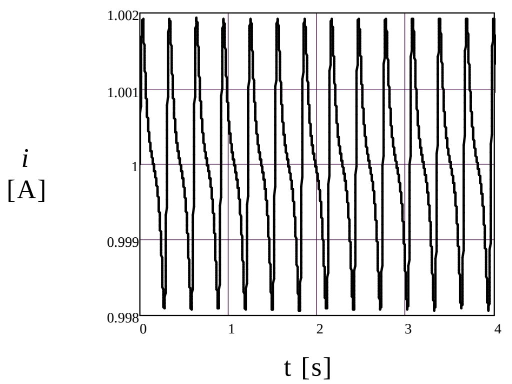

Fig. 6. Free undamped vibration - variation of i

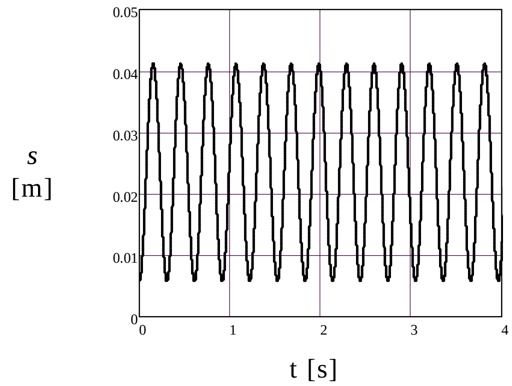

Fig. 7. Free undamped vibration - variation of s

The static equilibrium position has been found for $\varphi_0 = 0.819\,\mathrm{rad}$, that corresponds to a height of the plane air gap $x_0 = 0.023\,\mathrm{m}$

Figures 4-7 illustrate the variation of quantities φ, ω, i and s in the case of free undamped vibrations, i.e. when the damping was neglected.

The variation of the same quantities is illustrated in Figures 8-11 in the case of free damped vibrations, with a damping coefficient $c = 20$.

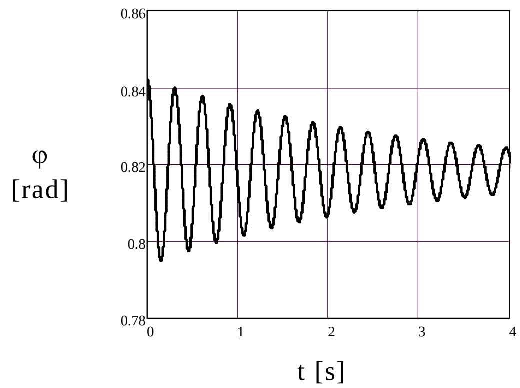

Fig. 8. Free damped vibration - variation of φ

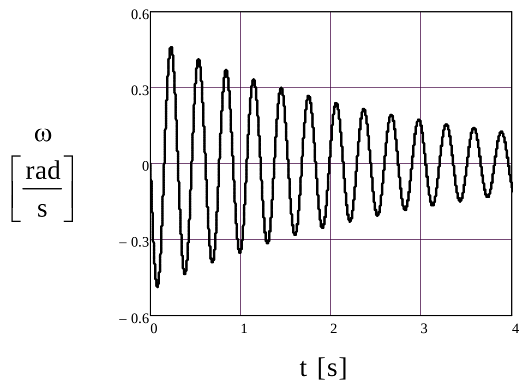

Fig. 9. Free damped vibration - variation of ω

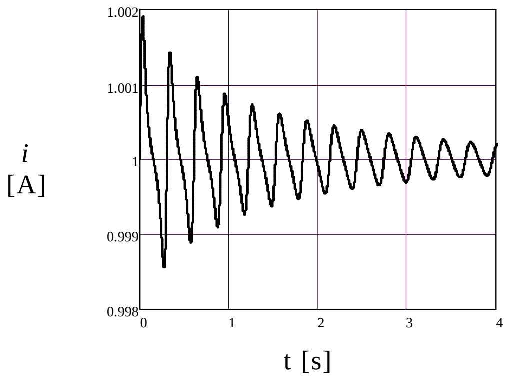

Fig. 10. Free damped vibration - variation of *i*

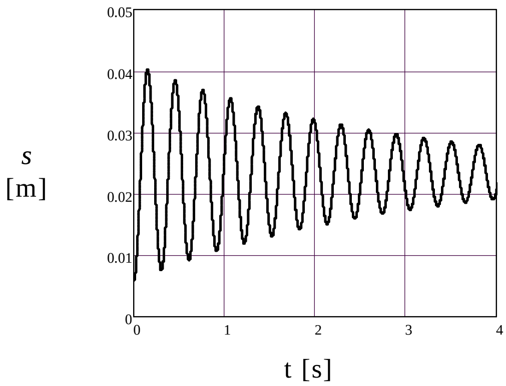

Fig. 11. Free damped vibration - variation of *s*

7. CONCLUSIONS

The analysis with Lagrange equations of the second species allows it to determine the differential equations of motion, taking into account both mechanical and electrodynamic characteristics of the electromechanical system proposed by the authors (Fig. 2).

Considering that the electrical subsystem has been subjected to a constant voltage, the obtained differential system has been studied, first, without and then with damping. For the adopted numerical values, the time variations of some characteristic quantities (angle φ, angular velocity ω, current i height s of the plane air gap) have been plotted. Despite the nonlinearity of the system, these quantities exhibit a periodic and a pseudoperiodic behavior, respectively.

Further studies should be dedicated to the case of the forced vibration, produced by an alternative voltage applied to the device.

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