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SHEAR STRESSES IN BEAMS WITH VARIABLE CROSS SECTION, SUBJECTED TO NON-UNIFORM BENDING

PhD. Assoc. Prof. Petru Dumitrache,
“Dunarea de Jos” University of Galati, Romania

ABSTRACT

The goal of this paper is to present a generalization of the Juravski formula (appropriate for the beams with constant cross section, subjected to non-uniform bending), for the beams with variable cross section.
In the first part of the paper are presented calculus hypotheses and is deducted the generalized formula and in second part, the new formula is applied for a few particular cases of beams with variable cross section and the results are verified using similar results from specialty literature and the element finite method. Also, some observations and comments are made.

KEYWORDS: shear stresses, non-uniform bending, variable cross section

1. PRELIMINARY

It is known that in the case of beams with constant cross section, subjected to non-uniform bending, the shear stresses are calculated using the Juravski formula.

For the Juravski formula deduction, besides assuming constancy of the beam cross section, are accepted Coulomb's hypothesis (independence of action of bending moment and shear force, [1]) and the hypothesis according to which the shear stress is constant along a parallel with neutral axis of the cross section.

For all of the following considerations, it is assumed that the forces acting on beam are located in xOy plan, Oy and Oz being axes of the beam cross section. Also, Oz axis is the neutral axis of the cross section. Finally, is denoted by h the height of the cross section and the length of the beam is denoted with l.

The Juravski formula is valid for regular beams ($l/h > 2$), in accordance with the classification of [1], in areas where are not applied concentrated loads and the variation of the shearing force may be neglected. It should be noted that the formula is used only for beams which have h/l ratio large enough not to be able to neglect the discrepancy between Bernoulli's hypothesis and the tendency of deviation from flatness of the cross-sections of the bended beam.

In accordance with the Juravski formula, the maximum shear stress τ_{xy} occurs at the neutral axis and is zero at both the top and bottom cross section of the beam.

In most courses of strength of materials it is deduced the Juravski formula, pointing out that if the cross-section of the beam is variable, one can apply the same formula and results have an acceptable level of error if changes of the cross section along the beam, are slow [3].

Furthermore we will deduct a more general formula than the Juravski formula and we will present some checks and conclusions regarding this new formula.

2. THE GENERALIZATION OF THE JURAVSKI FORMULA

Let us consider a beam which complies assumptions listed in the previous paragraph, with variable cross section. We examine the equilibrium in axial direction of a beam element, delimited by two transversal planes and a longitudinal plane, parallel to the neutral plane (see Fig. 1).

Be $d\Omega$, an element of area in cross section and y, the distance from it to the neutral axis. The equation of equilibrium of forces in axial direction of the beam element is:

$$\int_{\Omega+d\Omega} (\sigma + d\sigma) d\Omega - \int_{\Omega} \sigma d\Omega = \tau_{yx} b dx \quad (1)$$

Remark. The surface on which the shear stresses τ_{yx} are applied can be considered rectangular, without restrict generality, because in the calculations below, the higher-order small terms will be neglected.

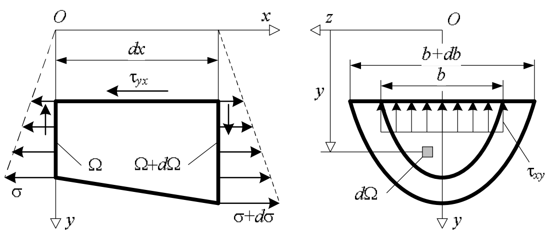

Fig. 1. Element of beam with variable cross-section

Are denoted by M_i and I_z the bending moment in the cross section current, respectively the axial second area moment of the cross section with respect to the neutral axis. It is also noted with S_z the static moment, with respect to neutral axis of the part in the current section, located below the longitudinal plane which includes the shear stresses τ_{yx}.

Taking into account the Navier's formula, with the above notations, equation (1) becomes

$$\left[\frac{M_i}{I_z} + d\left(\frac{M_i}{I_z}\right)\right] \int_{\Omega+d\Omega} y d\Omega - \frac{M_i}{I_z} \int_{\Omega} y d\Omega = \tau_{yx} b dx \quad (2)$$

Observing that the integrals in (2) are static area moments, we find:

$$\left[\frac{M_i}{I_z} + d\left(\frac{M_i}{I_z}\right)\right] (S_z + dS_z) - \frac{M_i}{I_z} S_z = \tau_{yx} b dx \quad (3)$$

After performing calculations in (3) and neglecting small higher-order terms, taking into account the principle of shear stresses parity, is obtained:

$$\tau_{xy} = \frac{S_z \left(\frac{dM_i}{dx} - \frac{M_i}{I_z} \cdot \frac{dI_z}{dx} \right) + M_i \frac{dS_z}{dx}}{b I_z} \quad (4)$$

The obtained formula allows for the calculation of the shear stresses τ_{xy} in cross sections of the beam subjected to non-uniform bending, the cross section of the beam being variable.

Remark. If the cross section is constant along the beam, the derivatives of the geometrical characteristics I_z and S_z are cancelled, resulting the Juravski formula.

Another important aspect is related to the value of the shear stresses τ_{xy} in the areas of the cross section which are located at the extreme distances from the neutral axis. In the case of the beams with constant cross section, these stresses are always zero because S_z is cancelled in the Juravski formula. As shown, in accordance with equation (4), for beams with variable cross section, shear stress τ_{xy} are no longer cancelled in the mentioned areas, but possibly for a particular location of the cross section in which the calculation is made .

Also, for beams with variable cross section, τ_{xy} maximum shearing stress does not occur in any section, in neutral axis, like in the case of the beams with constant cross section.

3. CASE STUDY

We propose to study the case of beams with variable rectangular cross section, the height of the section (h) being variable. In Figure 2 is shown an element of such a beam.

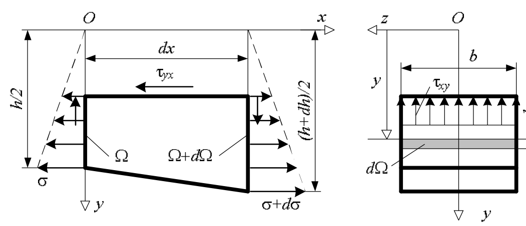

Fig. 2. Element of beam with variable rectangular cross-section

To obtain the formula for calculating the shear stress (τ_{xy}) is used the general formula (4), which, after a series of elementary transformations, can be written as:

$$\tau_{xy} \cdot b = S_z \frac{d}{dx}\left(\frac{M_i}{I_z}\right) + \frac{M_i}{I_z} \cdot \frac{dS_z}{dx} \quad (5)$$

Knowing that for the case considered, $I_z = bh^3/12$ and $S_z = b(h^2/4 - y)/2$, after calculations, we get:

$$\tau_{xy} = \frac{6}{b}\left(\frac{h^2}{4} - y^2\right) \frac{d}{dx}\left(\frac{M_i}{h^3}\right) + \frac{3M_i}{bh^2} \cdot \frac{dh}{dx} \quad (6)$$

We examine the case of a cantilever loaded with a concentrated force on the free end, with rectangular cross section with constant width (b) and linear variable height (h), according to the law:

$$h = h_0 \cdot \left(1 + k \cdot \frac{x}{l}\right) \qquad x \in [0,l],\ k > 0 \qquad (6)$$

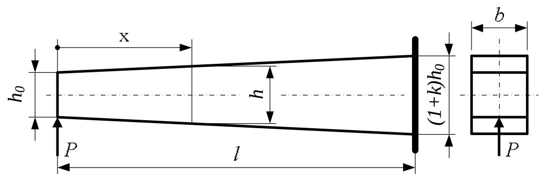

Fig. 3. Cantilever with linear variable rectangular cross-section

Since the bending moment is $M_i = Px$, based on relations (5) and (6), we obtain:

$$\tau_{xy} = \frac{6P}{hh_0\left(1 + k \cdot \frac{x}{l}\right)^2} \times \left[\frac{kx}{2l} + \frac{1}{h_0^2}\left(\frac{h^2}{4} - y^2\right)\frac{l(l - 2kx)}{(1 + kx)^2}\right] \qquad (7)$$

For $k = 1$ is obtained the particular case presented in [2].

Equation (7) allows us to analyze, in a certain section of the beam, how shear stress depends on the distance from the neutral axis. This analysis is possible by observing that in a certain section (x fixed), shear stress τ_{xy} is a function of degree 2, with variable y.

In this context, there are three cases:

a) $0 \le x < 0.5 \cdot l / k$ - shear stress varies with a parabola which has the maximum value in the neutral axis of the section. Shear stress has minimum values at both the top and bottom of the cross section.

b) $x = 0.5 \cdot l / k$ - shear stress does not depend on the variable y, parable inside brackets from (7) degenerating into a line parallel to the Oy axis.

c) $0.5 \cdot l / k < x \le l$ - shear stress varies with a parabola which has the minimum value in the neutral axis of the section. Shear stress has maximum values at both the top and bottom of the cross section.

It is important to note that the cross section located at distance $x = 0.5 \cdot l / k$ from the free end of the beam is the border between two areas in which the beam has an essentially different behaviour:

- Area where maximum shear stress has the maximum value in the neutral axis of the cross section.
- Area where the shear stress has maximum values at both the top and bottom of the cross section (area where is observed a pronounced deviation from the Juravski formula). In this case, considering that the normal stresses caused by bending have maximum values also at the extremities of the cross section, it follows that in this area of the beam the state of stresses in beam is more pronounced than it would result from the application of the Juravski formula.

The above considerations allow for an evaluation of the level of error produced by using the Juravski formula for beams with variable section. It is also observed that with the increasing of the coefficient k (the variation of the cross-section is more abrupt along the beam), the length of the area with pronounced deviation from the Juravski formula, is higher.

We will customize relation (7) for different values of the variable x, so we will find the laws of variation of shear stress τ_{xy}, depending on the distance y from the neutral axis. We will consider cross sections for which: $x = 0$, $x = 0.25l$, $x = 0.5l$ and $x = 0.75l$.

- For $x = 0$:

$$\tau_{xy} = \frac{1.5P}{bh_0} \cdot \left(1 - \frac{y^2}{h_0^2}\right) \qquad (8)$$

To the upper and lower extremities of the cross section ($y = \pm 0.5h_0$) shear stresses can be calculated with the following formula:

$$\tau_{xy} = 0 \qquad (9)$$

At the neutral axis ($y = 0$) shear stress can be calculated with formula:

$$\tau_{xy} = \frac{1.5P}{bh_0} \qquad (10)$$

- For $x = 0.25l$, $h = (1 + 0.25k)h_0$:

$$\tau_{xy} = \frac{6P}{bh_0(1+0.25\cdot k)^2} \times \left[0.125k + \frac{1}{h_0^2}\left(\frac{h^2}{4} - y^2\right)\frac{1-0.5k}{(1+0.25k)^2}\right] \quad (11)$$

To the upper and lower extremities of the cross section ($y = \pm 0.5(1+0.25k)h_0$) shear stresses can be calculated using formula:

$$\tau_{xy} = \frac{0.75k}{(1+0.25k)^2}\cdot\frac{P}{bh_0} \quad (12)$$

At the neutral axis ($y = 0$) shear stress can be calculated with formula:

$$\tau_{xy} = \frac{1.5}{(1+0.25\cdot k)^2}\cdot\frac{P}{bh_0} \quad (13)$$

- For $x = 0.5l$, $h = (1+0.5k)h_0$:

$$\tau_{xy} = \frac{6P}{bh_0(1+0.5k)^2} \times \left[0.25k + \frac{1}{h_0^2}\left(\frac{h^2}{4} - y^2\right)\frac{1-k}{(1+0.5k)^2}\right] \quad (14)$$

To the upper and lower extremities of the cross section ($h = \pm 0.5(1+0.5k)h_0$) shear stresses can be calculated using formula:

$$\tau_{xy} = \frac{1.5k}{(1+0.5k)^2}\cdot\frac{P}{bh_0} \quad (15)$$

At the neutral axis ($y = 0$) shear stress can be calculated with formula:

$$\tau_{xy} = \frac{1.5}{(1+0.5k)^2}\cdot\frac{P}{bh_0}. \quad (16)$$

- For $x = 0.75l$, $h = (1+0.75k)h_0$ and:

$$\tau_{xy} = \frac{6P}{bh_0(1+0.75k)^2} \times \left[0.375k + \frac{1}{h_0^2}\left(\frac{h^2}{4} - y^2\right)\frac{1-1.5k}{(1+0.75k)^2}\right] \quad (17)$$

To the upper and lower extremities of the cross section ($h = \pm 0.5(1+0.75k)h_0$) shear stresses can be calculated with the following formula:

$$\tau_{xy} = \frac{2.25k}{(1+0.75k)^2}\cdot\frac{P}{bh_0} \quad (18)$$

At the neutral axis ($y = 0$) shear stress can be calculated with formula:

$$\tau_{xy} = \frac{1.5}{(1+0.75k)^2}\cdot\frac{P}{bh_0} \quad (19)$$

- For $x = l$, $h = (1+k)h_0$ and:

$$\tau_{xy} = \frac{6P}{bh_0}\cdot\frac{1}{(1+k)^2} \times \left[0.5k + \frac{1}{h_0^2}\left(\frac{h^2}{4} - y^2\right)\frac{1-2k}{(1+k)^2}\right] \quad (20)$$

To the upper and lower extremities of the cross section ($h = \pm 0.5(1+k)h_0$) shear stresses can be calculated using the following formula:

$$\tau_{xy} = \frac{3k}{(1+k)^2}\cdot\frac{P}{bh_0} \quad (21)$$

At the neutral axis ($y = 0$) shear stress can be calculated using formula:

$$\tau_{xy} = \frac{1.5\cdot k}{(1+k)^2}\cdot\frac{P}{bh_0} \quad (22)$$

The numerical checking of the formulas previously obtained was done using the finite element method, considering the following numerical data: $k = 1$, $b = 20\ mm$, $h_0 = 20\ mm$, $l = 400\ mm$, $P = 10\ N$.

Pre-processing of the problem and post-processing of the numerical obtained results were performed using a FEA platform.

The beam was modelled using hexahedral parabolic finite elements in order to increase the accuracy of results. Also to ensure high accuracy of results has been opted for a high density mesh. In this context, the beam was modelled using 2256 finite elements, connected through 14,517 nodes.

To reduce the local effect generated by the mode of application of the force, it was distributed evenly over the nodes located along the neutral axis of the section located in the free end. The rigid fixing of the beam was modelled by removing all degrees of freedom of the nodes located in the section with maximum height.

In Figure 4 is shown a screen capture taken from FEA platform which presents the variation of the shear stresses both along the beam and the height of the cross sections. It is noted, first, a clear delimitation in the middle of the beam, (case $k = 1$) between the areas above discussed. In Figure 4, side from the right there is the area of the beam where a pronounced deviation from the Juravski's formula is.

Secondly, it is noted the local effect of boundary conditions (loading and fixing conditions). It is highlighted the fact that the above formulas do not "catch" these local effects.

Figure 5 presents a detail located near the half of the beam. In this figure are shown the nodes located on the axis Oy from the middle cross section of the beam.

Table 1 contains values of the shear tension from nodes shown in Figure 5 calculated with properly customized formulas (14), those found using finite element method and their relative errors. It is noted that the relative errors are very small, which justifies the above formulas.

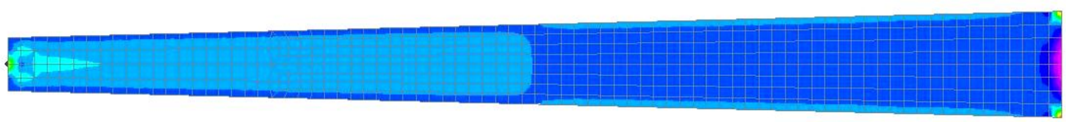

Fig. 4. Variation of the shear stresses

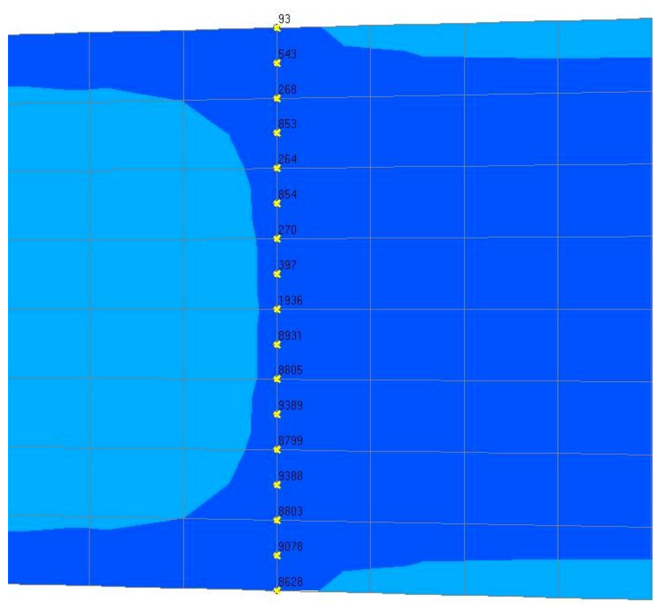

Fig. 5. Detail at the middle of the beam

Table 1

Node	τ_{xy} – formula (14) (N/mm^2)	τ_{xy} – FEA (N/mm^2)	Relative error (%)
8628	$1.(6)\cdot 10^{-2}$	$1.66714\cdot 10^{-2}$	0.0284
9078		$1.66636\cdot 10^{-2}$	-0.0184
8803		$1.66558\cdot 10^{-2}$	-0.0652

9388		$1.66565 \cdot 10^{-2}$	-0.0610
8799		$1.66572 \cdot 10^{-2}$	-0.0568
9389		$1.66608 \cdot 10^{-2}$	-0.0352
8805		$1.66644 \cdot 10^{-2}$	-0.0136
8931		$1.66672 \cdot 10^{-2}$	-0.0032
1936		$1.66698 \cdot 10^{-2}$	0.0188
397		$1.66672 \cdot 10^{-2}$	-0.0032
270		$1.66644 \cdot 10^{-2}$	-0.0136
854		$1.66608 \cdot 10^{-2}$	-0.0352
264		$1.66572 \cdot 10^{-2}$	-0.0568
853		$1.66565 \cdot 10^{-2}$	-0.0610
268		$1.66558 \cdot 10^{-2}$	-0.0652
543		$1.66636 \cdot 10^{-2}$	-0.0184
93		$1.66714 \cdot 10^{-2}$	0.0284

4. CONCLUSIONS

The use of the Juravski's formula for beams with variable cross section subjected to non-uniform bending may lead to unacceptable errors, especially in case of abrupt variation. The generalized formula (4) can be applied, properly customized, in many practical situations and allows for an accurate assessment of the state of stress (obviously, within the assumptions imposed) that is developed in such beams subjected to non-uniform bending.

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