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ON THE USE OF APPROXIMATE NUMERICAL AND ANALYTICAL METHODS IN ORDER TO SOLVE THE DIFFERENTIAL EQUATIONS THAT DESCRIBE VIBRATIONS OF THE MECHANICAL SYSTEMS

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ABSTRACT

The paper makes a comparative study on the approximation errors of the solutions of the differential equations of second order using different numerical methods, a study which is assessed by the necessity of numerical solving of the non-linear differential equations that describe the vibrations of the mechanical systems. For every resulted analytical solution from a differential equation we can compare the approximation errors for movement, speed and acceleration using the Runge-Kutta and finite differences methods. Then, the smallest approximation errors of the numerical method will be compared to the approximation errors using a linearization method that the author published in a previous paper. In the end, we present conclusions and recommendations concerning the use of approximation numerical methods for the approximate solutions.

KEYWORDS: mechanical system, non-linear vibration

1. INTRODUCTION

Forced vibrations of the harmonical oscillators are described by second order differential equations such as

$$m\ddot{x} + h\dot{x} + kx = F \cdot \sin \omega t \qquad (1)$$

where x is the movement, m is the oscillators mass, h is the damping factor, k is the elastic constant and $F \cdot \sin \omega t$ is the excitation force.

The analytical solution of the differential equation is

$$x = e^{-n \cdot t} \cdot a \cdot \sin(p_1 \cdot t + \phi) + A \cdot \sin(\omega \cdot t - \alpha) \qquad (2)$$

where

$$p_1 = \sqrt{p^2 - n^2}\ ,\ p^2 = \frac{k}{m},\ 2n = \frac{h}{m}. \qquad (3)$$

and φ and α are the initial phases of the free respectively the damped oscillation, a and A result from the initial conditions of movement.

In [1] it is presented an approximate analytical method for solving the differential equation (1) using linearization, when k is not a constant but depends on the movement through a polynomial expression

$$k = k_0 + ax + bx^2 + cx^3 + dx^4 + \ldots \qquad (4)$$

where k_0 is a rigidity static coefficient and $a,b,c,d\ldots$ are rigidity coefficients in a dynamic regime.

We obtain the approximate analytical solution

$$x(t) = w_1 \sin(\omega t + \varphi_1) + w_2 \sin(2\omega t + \varphi_2) + w_3 \sin(3\omega t + \varphi_3) + w_4 \sin(4\omega t + \varphi_4) + w_5 \sin(5\omega t + \varphi_5) + \frac{z_0}{p^2} \qquad (5)$$

where

$$w_k = \frac{z_k}{(p^2 - k^2\omega^2)^2 + 4k^2n^2\omega^2} \cdot Y,$$
$$Y = \sqrt{k^2\omega^2(4n^2+1) - p^2}$$

$$\varphi_k = \begin{cases} arctg \dfrac{2kn\omega}{k^2\omega^2 - p^2} & ,k\ odd \\ arctg \dfrac{p^2 - k^2\omega^2}{2kn\omega} & ,k\ even \end{cases},$$
$$k = 1,2,....,5 \qquad (6)$$

The sizes $w_1, w_2, ..., w_5$ are amplitudes and $\varphi_1, \varphi_2, ..., \varphi_5$ are the phases of the excitation spectrum.

2. THE EVALUATION OF THE APPROXIMATION ERRORS

We will do a comparative study on the approximation errors of the solutions from the differential equation (1), comparing the results obtained by numerical integration methods (Runge-Kutta, finite differences) to the exact analytical solution and the one obtained through linearization. The maximum relative misbehaviour is calculated

$$\xi^{RK}{}_i = \frac{x_{RK_i} - x_{AN_i}}{x_{MAX\,AN} - x_{MIN\,AN}},$$

$$\xi^{DF}{}_i = \frac{x_{DF_i} - x_{AN_i}}{x_{MAX\,AN} - x_{MIN\,AN}}, i = 1,2,... \qquad (7)$$

where

x_{RK_i} - the movement calculated with Runge-Kutta method at t_i moment;

x_{DF_i} - the movement calculated with finite differences at t_i moment;

x_{AN_i} - the movement calculated from the analitical solution from the t_i moment;

$x_{MAX\,AN}$, $x_{MIN\,AN}$ - the highest and the lowest value of movement, calculated with analytical solution.

The quadratic medium misbehaviour will be

$$\xi = \sqrt{\frac{\sum_{i=1}^{n} \xi_i^2}{n}}, \qquad (8)$$

n being the number of moments when we make the comparisons.

3. NUMERICAL EXAMPLES

At the beginning we shall see the misbehaviours from the exact analytical solution when we change the step of integration.

The numerical data for the differential equation (1) are:

$$m = 600kg; \quad h = 864\ N \cdot s/m;$$
$$k = 2 \cdot 10^6\ N/m; \quad F = 2 \cdot 10^4\ N;$$
$$\omega = 50\pi\ rad/s$$

and the initial conditions are null.

In figure 1, which shows the graphical solution, we observe the evolution of errors from the two numerical methods for the increment $dt = 10^{-3}$ seconds.

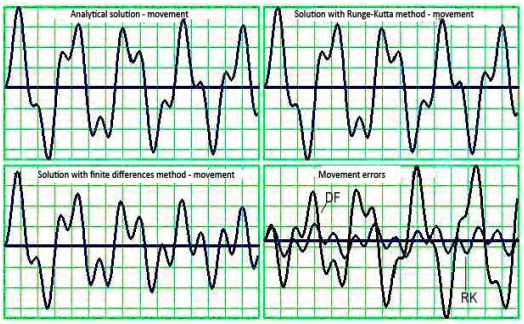

Fig.1 Errors when approximating movement for an elastic linear system in a forced vibration, with the increment $dt = 10^{-3}$ s.

For duration 0.50s it results: movement between 0.006 and -0.005 m; maximum errors between 18.705 and -19.374%; medium quadratic misbehaviour using Runge-Kutta method 2.100; medium quadratic misbehaviour using finite differences method 9.434.

In figure 2, which shows the graphical solution, we observe the evolution of errors from the two numerical methods for the increment $dt = 10^{-4}$ seconds.

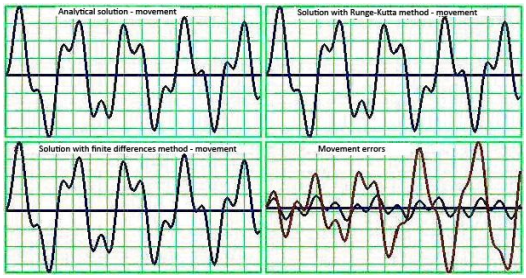

Fig.2 Errors when approximating movement for an elastic linear system in a forced vibration, with the increment $dt = 10^{-4}$ s.

For duration 0.50s it results: movement between 0.006 and -0.005 m; maximum errors between 2.293 and -2.266%; medium quadratic misbehaviour using Runge-Kutta method 0.210; medium quadratic misbehaviour using finite differences method 1.086.

From the error analysis, we show that for both methods we find a reduction of errors when we lower the time increment and a better precision of the Runge-Kutta method than the finite differences method. We find out that the level of precision characterized by the medium quadratic misbehaviour increases directly proportional with the decrease of the increment.

When we cannot find an exact analytical solution for a differential equation we obtain the solution using the numerical way. If the solution is a movement, we find other laws of oscillatory movement like speed and acceleration through numerical specific calculation such as numerical derivation of space or integration of the differential equation after a modified algorithm in order to obtain speed and acceleration. We shall do a numerical solving to observe the level of errors for speed and acceleration.

In figures 3, 4 and 5 we compare the solutions of the differential equation (1) for an increment $dt = 10^{-5}$ during $0{,}5$ seconds, but this is done for free damping oscillations (homogeneous equation).

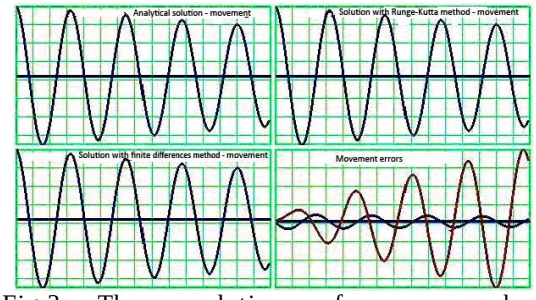

Fig.3 The evolution of errors when approximating movement for an elastic linear system in a free damped vibration.

For duration 0.50s it results: movement between 0.982 and -0.961 m; maximum errors between 0.295 and -0.272%; medium quadratic misbehaviour using Runge-Kutta method 0.018; medium quadratic misbehaviour using finite differences method 0.138.

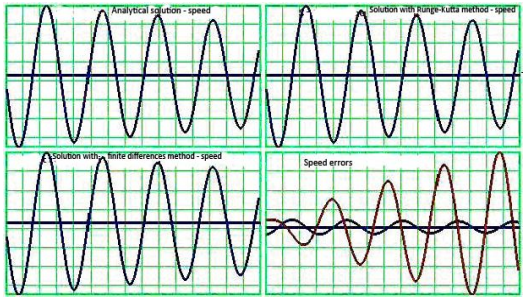

Fig.4 The evolution of errors when approximating speed for an elastic linear system in a free damping vibration.

For duration 0.50s it results: speed between 54.443 and -56.613 m/s; maximum errors between 0.287 and -0.263 %; medium quadratic misbehaviour using Runge-Kutta method 0.018; medium quadratic misbehaviour using finite differences method 0.134.

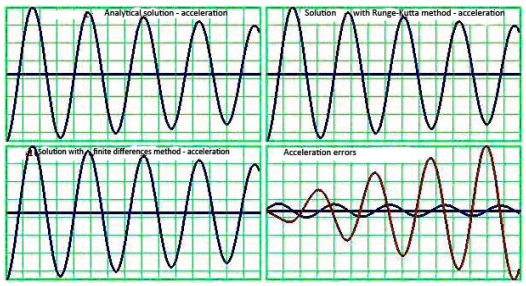

Fig.5 The evolution of errors when approximating acceleration for an elastic linear system in a free damping vibration.

For duration 0.50s it results: acceleration between 3203.951 m/s/s and -3256.019 m/s/s; maximum errors between 0.273 and -0.295%; medium quadratic misbehaviour using Runge-Kutta method 0.018; medium quadratic misbehaviour using finite differences method 0.139. When observing the misbehaviours in the approximation of speed and acceleration using the two numerical methods we see the absence of propagation of the approximation errors, sometimes the level of errors improves when we calculate the approximate speed or acceleration rather than obtaining the space.

The examples we presented have analysed the phenomenon for a small increment of time. In order to understand more, we exam the misbehaviours for a bigger increment of time, of 4 seconds, figures 6, 7 and 8.

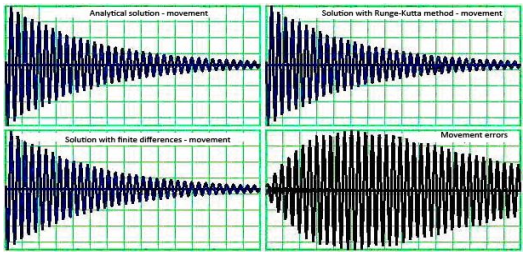

Fig.6 The evolution of errors when approximating movement for an elastic linear system in a free damping vibration.

For duration 4.00s it results: movement between 0.917 and -0.960 m; maximum errors between 0.449 and -0.448%; medium quadratic misbehaviour using Runge-Kutta method 0.009; medium quadratic misbehaviour using finite differences method 0.244.

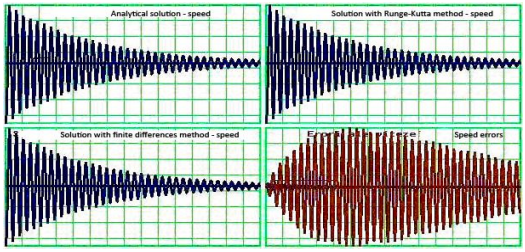

Fig.7 The evolution of errors when approximating speed for an elastic linear system in a free damping vibration.

For duration 4.00s it results: speed between 54.267 and -56.613 m/s; maximum errors between 0.438 and -0.439%; medium quadratic misbehaviour using Runge-Kutta method 0.009; medium quadratic misbehaviour using finite differences method 0.239.

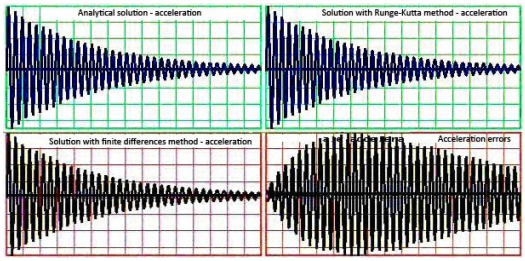

Fig.8 The evolution of errors when approximating acceleration for an elastic linear system in a free damping vibration.

For duration 4.00s it results: acceleration between 3203.951 ms/s/s and -3067.422 m/s/s; maximum errors between 0.447 and -0.447%; medium quadratic misbehaviour using Runge-Kutta method 0.009; medium quadratic misbehaviour using finite differences method 0.244. We observe that for solutions calculated on a bigger increment of time, the level of errors decreases when applying the Runge-Kutta method and increases when applying the finite differences method, especially when we analyse speed and acceleration. Back to forced oscillation, the calculation errors of movement, speed and acceleration will have the values from figures 9, 10 and 11, using an increment $dt = 10^{-5}$.

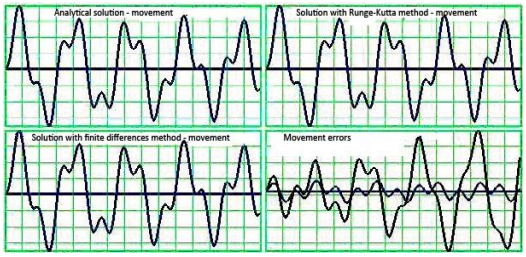

Fig.9 The evolution of errors when approximating movement for an elastic linear system harmonically excited.

For duration 0.50s it results: movement between 0.006 and -0.005 m; maximum errors between 0.234 and -0.230%; medium quadratic misbehaviour using Runge-Kutta method 0.021; medium quadratic misbehaviour using finite differences method 0.110.

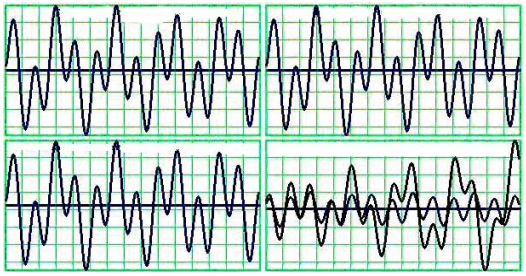

Fig.10 The evolution of errors when approximating speed for an elastic linear system harmonically excited.

For duration 0.50s it results: speed between 0.452 and -0.461 m/s; maximum errors between 0.224 and -0.202%;medium quadratic misbehaviour using Runge-Kutta method 0.031; medium quadratic misbehaviour using finite differences method 0.091.

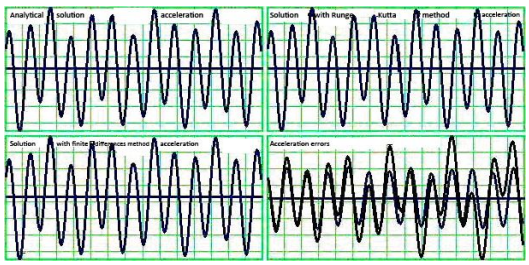

Fig.11 The evolution of errors when approximating acceleration for an elastic linear system harmonically excited.

For a duration of 0.50s it results: acceleration between 50.113 and -52.151 m/s/s; maximum errors between 0.132 and -0.131%; medium quadratic misbehaviour using Runge-Kutta method 0.042; medium quadratic misbehaviour using finite differences method 0.061.

We observe that in the case of forced oscillations we don't notice esential differences when comparing to the case of free damping oscillations.

The analysis above was made for a system that produces free or forced oscillations with a variation for one excitation of pulsation.

For a full examination it is necessary to find out the evolution of the calculation errors compared to the variation for the excitation of pulsation.

The errors are calculated using the values of the amplitudes resulted for different frequences, compared to the amplitude given by the exact solution with the time being 0,5s or 1s as in figure 12.

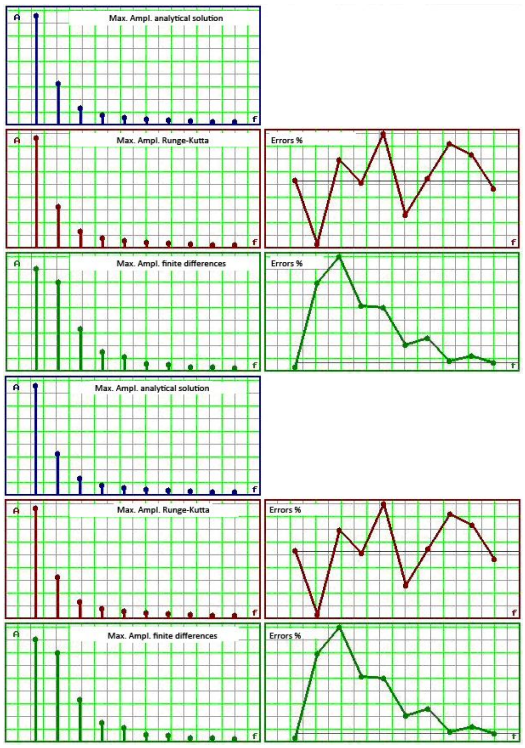

Fig.12 The evaluation of errors when approximating movement using Runge-Kutta and finite differences methods for an elastic linear system harmonically excited for a duration of 0,5s and 1s.

For a duration of 0.50s it results: medium quadratic misbehaviour using Runge-Kutta method is 0.0013; relative errors between -0.00263% and 0.00193%; medium quadratic misbehaviour using finite differences method =0.7098; relative errors between -0.06514 and 1.50515%; frequency 25Hz.

For a duration of 1.00s it results: medium quadratic misbehaviour using Runge-Kutta method is 0.0013; relative errors between -0.00263 and 0.00193%; medium quadratic misbehaviour using finite differences method 0.7098; relative errors between -0.06514 and 1.50515%; frequency 25Hz.

Further more we will compare the approximation errors when obtaining the analytical approximate solution using linearization with the one obtained using Runge-Kutta method, which proved to be more precise.

For the experimental data presented above:

$$a = -1{,}25 \cdot 10^{7}\ N/m^{2},\ b = -1{,}58 \cdot 10^{-6}\ N/m^{3}$$

$$c = 7{,}13\ N/m^{4},\ d = 5{,}4 \cdot 10^{-12}\ N/m^{5},$$

$$k_0 = 2 \cdot 10^{6}\ N/m,\ A = 8 \cdot 10^{-4}\ m,$$

we obtain the results provided in figure 13.

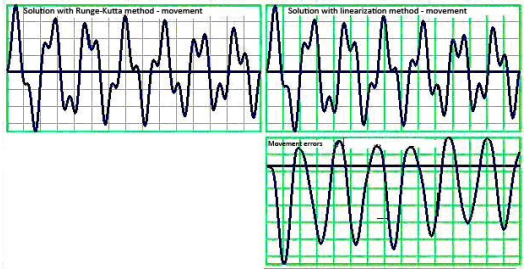

Fig. 13. The evaluation of the error level when approximating movement using the linearization method.

For a duration of 0.75s it results: movement between 5.701 and -5.216 m; maximum errors between 0.484 and -1.694%; medium quadratic misbehaviour 0.711.

Increasing the non-linearization grade while changing the values of coefficients a,b,c,d, the level of approximation does not change significantly.

Analysing the misbehaviours, we need to calculate the absolute error compared to the exact solution, using the solution obtained through linearization.

If the exact solution from the Runge-Kutta method has a ε_1 quadratic medium misbehaviour compared to the exact solution, and the solution obtained using linearization has a ε_2 quadratic medium misbehaviour compared to the one from the Runge-Kutta method, the quadratic medium misbehaviour of the solution obtained through linearization

compared to the exact solution will be in the worst case $\varepsilon_1 + \varepsilon_2$ which is acceptable.

The solution obtained using the linearization looks less precise than the ones resulted from the numerical methods, which could be a disadvantage. Despite this incovenient, which can be corrected, the method offers a great advantage: the posibility of obtaining an analytical solution, even though it is be an approximate one.

In the linearization process we start from a solution at first approximation of the following form

$$x = A \cdot \sin \omega t \qquad (9)$$

with the amplitude A known. For the method to be appliable in any situation, we must find a way to find out this amplitude.

In order to solve this problem:

- we solve the differential equation that characterised the system's vibrations using the Runge-Kutta method and we show a maximum amplitude A_{max};
- we solve the equation using the linearization starting with the solution with amplitude A given by the previous solving.

We need to specify that we cannot work with random values for the amplitude A because the results will be severely affected by errors.

Using a numerical simulation with the anterior data and $F = 100N$ and applying the Runge-Kutta method with a value $A_{max} = 2{,}9 \cdot 10^{-3}\ m$, the quadratic medium misbehaviour is a lot lower than when we use an experimental value for the amplitude for the solution at the first approximation.

4. CONCLUSIONS

Analysing results, we observe that the Runge-Kutta method offers a better precision when approximating solutions, compared to the finite differences method, erros being with at least one order lower, for a better precision we need to use a small increment.

The linearization method has limitations not only to smaller grades of non-linearity, but its adaptation to any concrete situations will be done by choosing a larger or smaller number of terms in the polynom that describes the coefficient of elasticity. Choosing correctly the movement's amplitude in the first approximation means that the level of errors is comparable to the Runge-Kutta method, also offering the advantage of an analytical research of the non-linear vibrations of the system with different generalizations being possible.

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