THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
FASCICLE XIV MECHANICAL ENGINEERING, ISSN 1224-5615
2011

CONSIDERATIONS REGARDING THE STABILITY OF HIGH-SPEED SHAFTS

PhD. Assoc. Prof. Cristian DRAGOMIRESCU
"Politehnica" University of Bucharest,
Dept. of Mechanics,
PhD. Assoc. Prof. Andrei CRAIFALEANU
"Politehnica" University of Bucharest,
Dept. of Mechanics,
PhD. Prof. Victor ILIESCU
Technical University of Civil Engineering of Bucharest, Dept. of Mechanics, Statics and Dynamics of Structures

ABSTRACT

The paper studies the problem of the dynamic stability of a high-speed shaft, subjected to bending vibrations. The case of normal machine tool shafts placed on rigid or flexible bearings is considered for different situations. Specific methods of the theory of dynamic system are used. The computer simulation results are interpreted.

KEYWORDS: rotor, stability, vibrations

1. INTRODUCTORY NOTIONS

The paper analyzes circular cross-section shafts, supported by flexible (elastic or elastoplastic) bearings, with different properties in horizontal and vertical planes. Particular cases of bearing properties, which affect the rotation motion, are studied from the stability point of view.

The analyzed system consists in a disk symmetrically positioned on the shaft, which is supported by flexible bearings with different characteristics (Fig. 1). The mass of the shaft is considered negligible with respect to the mass of the disk.

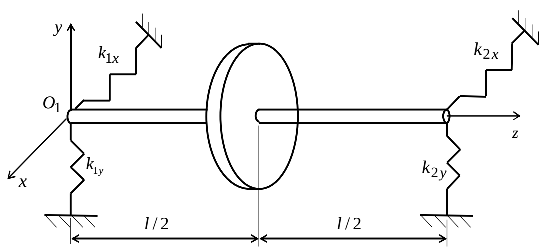

Fig. 1. High-speed shaft

The differential equations of motion are obtained, for the undamped bending vibrations [1], taking into account the gyroscopic effects of the disk.

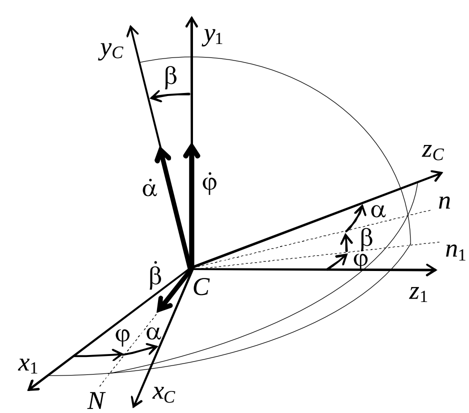

Fig. 2. Reference systems and Euler angles

These equations are determined by means of Lagrange equations of the second species [5], which are particularized according to the elastic characteristics of the bearings.

The position of the disk in space is

determined with respect to the fixed reference frame O_1xyz and it is specified by the coordinates x_C and y_C of the mass centre C (assuming that C remains in the same transverse plane, during the vibration) and by Euler angles α, β, φ (Fig. 2), which determine the following motions:

- precession, with angular velocity $\dot{\varphi}$;
- nutation, with angular velocity $\dot{\beta}$;
- spin, with angular velocity $\dot{\alpha}$.

Both movable reference frames, $Cx_1y_1z_1$ and $Cx_Cy_Cz_C$, have the origins in the mass centre. Reference frame $Cx_1y_1z_1$ has the axes parallel to the corresponding ones of O_1xyz, while $Cx_Cy_Cz_C$ is attached to the disk, with axis Cz_C perpendicular to it.

2. DIFFERENTIAL EQUATIONS OF MOTION

The differential equations which describe the vibrations of the disk-shaft system, that rotates uniformly with angular velocity ω, is [3]

$$\begin{cases} m\ddot{x} + \gamma_1 x - \delta_1\alpha = me\omega^2\cos(\omega\cdot t) \\ m\ddot{y} + \gamma_2 y - \delta_2\beta = me\omega^2\sin(\omega\cdot t) \\ J\ddot{\alpha} + J_{z_C}\omega\dot{\beta} - \gamma_1\dfrac{l^2}{4}\alpha - \delta_1 x = 0 \\ J\ddot{\beta} - J_{z_C}\omega\dot{\alpha} + \gamma_2\dfrac{l^2}{4}\beta - \delta_2 y = 0, \end{cases} \tag{1}$$

with

$$\begin{cases} \gamma_1 = c_{11}\cdot \\ \quad\cdot\left[1 - \dfrac{c_{11}}{2}\left(\dfrac{1}{c_{11}+2k_{1x}} + \dfrac{1}{c_{22}+2k_{2x}}\right)\right] \\ \gamma_2 = c_{11}\cdot \\ \quad\cdot\left[1 - \dfrac{c_{11}}{2}\left(\dfrac{1}{c_{11}+2k_{1y}} + \dfrac{1}{c_{22}+2k_{2y}}\right)\right] \\ \delta_1 = \dfrac{c_{11}^2 l}{4}\left(\dfrac{1}{c_{11}+2k_{1x}} + \dfrac{1}{c_{22}+2k_{2x}}\right) \\ \delta_2 = \dfrac{c_{11}^2 l}{4}\left(\dfrac{1}{c_{11}+2k_{1y}} + \dfrac{1}{c_{22}+2k_{2y}}\right), \end{cases} \tag{2}$$

$$\begin{cases} c_{11} = \dfrac{b_{22}}{a_{11}b_{22} - a_{12}b_{12}} \\ c_{22} = \dfrac{a_{11}}{a_{11}b_{22} - a_{12}b_{12}}, \end{cases} \tag{3}$$

where the following notations have bee used:

x, y – displacements of the fixing point of the disk on the shaft;

e – eccentricity of the mounting of the disk on the shaft;

m – mass of the disk;

$k_{1x}, k_{2x}, k_{1y}, k_{2y}$ – elasticity coefficients of the two bearings, with respect to axes O_1x and O_1y, respectively;

$a_{11}, b_{22}, a_{12} = b_{12}$ – static influence coefficients.

The determination of the differential equation system of the bending vibration of high-speed shafts, taking into account the gyroscopic effect, is presented in detail in reference [2].

The stability of system (1) has been studied for certain initial conditions and numerical values, in two particular cases:

a) shafts supported by rigid bearings,

$$\begin{cases} k_{1x} = k_{2x} = k_{1y} = k_{2y} \\ \gamma_1 = \gamma_2 = c_{11} \\ \delta_1 = \delta_2 = 0; \end{cases} \tag{4}$$

b) shafts supported by bearings with the same rigidity on transverse directions,

$$\begin{cases} k_{1x} = k_{2x} = k_1, \quad k_{1y} = k_{2y} = k_2 \\ \gamma_1 = \dfrac{2k_1c_{11}}{c_{11}+2k_1}, \quad \gamma_2 = \dfrac{2k_2c_{11}}{c_{22}+2k_2} \\ \delta_1 = \delta_2 = 0. \end{cases} \tag{5}$$

For such cases, the attractors have been found, i.e. the stable limit cycles on which the representative point of the motions of the system will be situated [4].

3. STABILITY OF SHAFTS IN PARTICULAR CASES

3.1. Shafts supported by rigid bearings

If the shaft is supported by rigid bearings, system (1) is equivalent with two independent systems [1]:

$$\begin{cases} m\ddot{x} + c_{11}x = me\omega^2 \cos(\omega \cdot t) \\ m\ddot{y} + c_{11}y = me\omega^2 \sin(\omega \cdot t), \end{cases} \tag{6}$$

$$\begin{cases} J\ddot{\alpha} + J_{z_C}\omega\dot{\beta} - c_{11}\frac{l^2}{4}\alpha = 0 \\ J\ddot{\beta} - J_{z_C}\omega\dot{\alpha} + c_{11}\frac{l^2}{4}\beta = 0. \end{cases} \tag{7}$$

Since equations (7) describe the free vibrations of the disk about its mass centre, they do not lead to other critical angular velocity than that induced by the motion described by (6). This system has the solution

$$\begin{cases} x = \dfrac{e\omega^2}{\frac{c_{11}}{m} - \omega^2}\cos(\omega \cdot t) \\ y = \dfrac{e\omega^2}{\frac{c_{11}}{m} - \omega^2}\sin(\omega \cdot t), \end{cases} \tag{8}$$

which corresponds to a circular closed trajectory, described by the equation

$$x^2 + y^2 = \left(\frac{e\omega^2}{\frac{c_{11}}{m} - \omega^2} \right)^2. \tag{9}$$

3.2. Shafts supported by bearings with the same rigidity on transverse directions

Similar to the previous particular case, if the shaft is supported by bearings with the same rigidity on transverse directions, system (1) is equivalent with two independent systems:

$$\begin{cases} m\ddot{x} + \dfrac{2k_1c_{11}}{c_{11} + 2k_1}x = me\omega^2\cos(\omega \cdot t) \\ m\ddot{y} + \dfrac{2k_2c_{11}}{c_{22} + 2k_2}y = me\omega^2\sin(\omega \cdot t), \end{cases} \tag{10}$$

$$\begin{cases} J\ddot{\alpha} + J_{z_C}\omega\dot{\beta} - \dfrac{2k_1c_{11}}{c_{11} + 2k_1}\dfrac{l^2}{4}\alpha = 0 \\ J\ddot{\beta} - J_{z_C}\omega\dot{\alpha} + \dfrac{2k_2c_{11}}{c_{22} + 2k_2}\dfrac{l^2}{4}\beta = 0. \end{cases} \tag{11}$$

System (10) has the solution

$$\begin{cases} x = \dfrac{e\omega^2}{\frac{2k_1c_{11}}{m(c_{11} + 2k_1)} - \omega^2}\cos(\omega \cdot t) \\ y = \dfrac{e\omega^2}{\frac{2k_2c_{11}}{m(c_{22} + 2k_2)} - \omega^2}\sin(\omega \cdot t), \end{cases} \tag{12}$$

which corresponds to an elliptically closed trajectory, described by the equation

$$\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} = 1, \tag{13}$$

where

$$\begin{cases} a_x = \dfrac{2k_1c_{11}}{m(c_{11} + 2k_1)} - \omega^2 \\ a_y = \dfrac{2k_2c_{11}}{m(c_{22} + 2k_2)} - \omega^2. \end{cases} \tag{14}$$

By plotting the trajectories in the phase plane, it can be shown that these trajectories are spirals, which start from a point inside or outside the ellipse, according to the initial conditions.

In each case, the attractor basin is the whole phase plane and the ellipse corresponding to the periodical solution is a limit cycle attractor, which is characteristic to elastic systems acted by periodical forces [4].

4. NUMERICAL RESULTS

Representations in the phase plane have been made (Fig. 3-5) for different values of the coefficients of the general form [1]

$$\ddot{x} = -ax + b\cos(\omega \cdot t), \tag{15}$$

all of them considering the initial conditions

$$t = 0 \quad \Rightarrow \quad x = x_0, \quad \dot{x} = v_0. \tag{16}$$

The second order diferential equation (15) is equivalent to the first order diferential system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1 + b\cos(\omega \cdot t), \end{cases} \tag{15'}$$

which is necessary in order to perform the numerical integrations that provided the above mentioned representions.

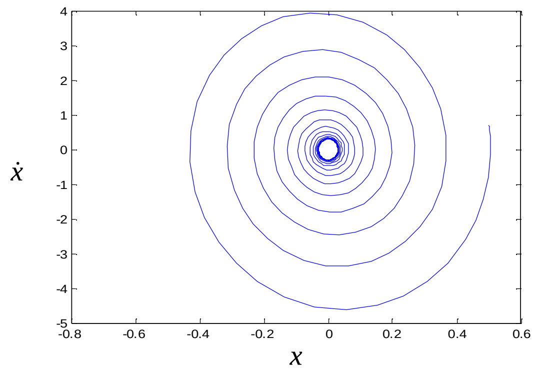

Fig. 3. $a = 97.87847$, $b = 0.30277$, $\omega = 9.8696$

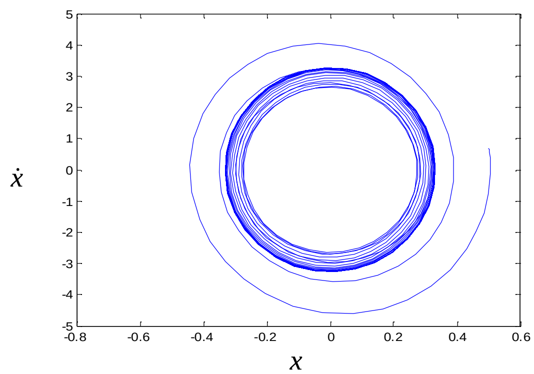

Fig. 4. $a = 103.32380$, $b = 3.81514$, $\omega = 9.8696$

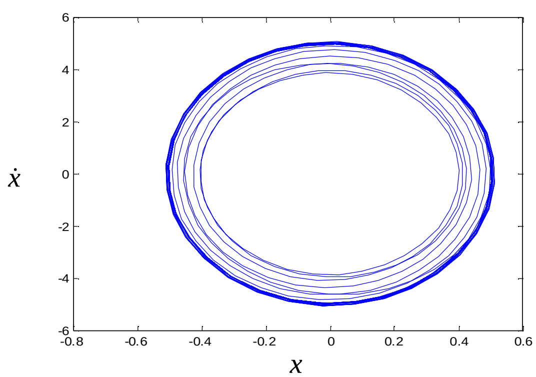

Fig. 5. $a = 109.81750$, $b = 8.00379$, $\omega = 9.8696$

If the shaft is supported by rigid bearings, the portrait in the phase plane, corresponding to equation (9), is circular, the angular velocity has the expression

$$\omega_{cr} = \sqrt{\frac{c_{11}}{m}}, \qquad (17)$$

while the precession is synchronous and direct, with self centering effect as ω increases.

If the shaft is supported by bearings with the same rigidity on transverse directions, there exist two critical angular velocities,

$$\omega_1 = \sqrt{\frac{\gamma_1}{m}}, \quad \omega_2 = \sqrt{\frac{\gamma_2}{m}}, \qquad (18)$$

while the motion of the center of the shaft is composed of a direct and an inverse circular precession.

For

$$\omega = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}}, \qquad (19)$$

the motion becomes a pure inverse precession.

5. CONCLUSIONS

Starting from the mathematical model described by differential equation system (1), corresponding to bending vibrations of high-speed shafts, by taking into account the gyroscopic effect and by considering some particular cases frequently used in machine design, the limit cycle attractor has been found, for the chosen initial conditions and numerical values.

Diagrams in the phase plane have been obtained by integrating the system of differential equations of motion.

For the considered values, the elliptical shape of the representations has been shown, calculation relationships of the critical angular velocities were given, and the conditions for inverse precession have been determined.

The analysis of periodical motions is very important, since it separates the stable and the unstable configurations of a dynamic system. Consequently, the conditions that parameters of the system should fulfill for periodical motion, when nonlinear effect are taken into account, are expected to lead also to the identification of varieties in the control space (the space of the parameters) which separate the stable and the unstable domains.

REFERENCES

[1] **Deciu, E., Dragomirescu, C.**, *Maschinendynamik*, Ed. Printech, Bucureşti, 2001.

[2] **Ion, C., Ion, E., Iliescu, V.**, *Contribuţii la studiul vibraţiilor la încovoiere a arborilor de turaţie foarte mare*, Acta Universitatis CIBINIENSIS, vol. XVII, Seria Tehnică, C. Mecanică aplicată, Sibiu, 1995.

[3] **Moraru, V., Ispas, C., Rusu, S.**, *Vibraţiile şi stabilitatea maşinilor-unelte*, Ed. Tehnică, Bucureşti, 1982.

[4] **Voinea, R., Store, I.**, *Introducere în teoria sistemelor dinamice*, Ed. Academiei Române, Bucureşti, 2000.

[5] **Voinea, R., Deciu, E., Dragomirescu, C.**, *Technische Mechanik*, Ed. ALMA, Craiova, 2009.