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ON THE STRESSES AND DEFLECTIONS CALCULUS IN BENDING THE CYLINDRICAL NATURALLY TWISTED BARS, WITH CONSTANT PITCH

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ABSTRACT

The present paper deals with some aspects of the stresses and deflections calculus in case of the cylindrical, naturally twisted bars, with constant pitch, on the basis of assumptions on the linear-elastic behavior of the material and on the small bending deflections.
In the first section of the paper, the main aspects concerning the geometry generation and classification of the naturally twisted bars are taken into account. Furthermore, the generalized formulas for the above-mentioned calculus are presented. Also, some aspects of the generalized formulas used are provided.
In last section of the paper, on the basis of the generalized formulas, a case study and conclusions are presented.

KEYWORDS: naturally twisted bar, stresses, deflections

1. Preliminary

The naturally twisted bars, known as pre-twisted bars, or helical rods, are structural elements with geometry generated from the helical movement of a generator plan domain. The axis of the helical movement intersects generating domain and is perpendicular to it. In this context, any point of the generating plan domain, except for the point of intersection with the axis of the helical movement, generates a no degenerate helical curve.
In general, naturally twisted bars can be classified according to the *geometric shape of the generating domain*, depending on *the position of the axis of the helical movement about the centre of gravity of the generating domain*, depending on *the nature of the no degenerate helical curves* and depending on *the pitch of the helical curves*.
Considering the *geometric shape of the generating domain* (which determines the cross section of the bar), the naturally twisted bars can be with rectangular cross section, with circular cross section, etc.
Taking into account *the position of the axis of the helical movement about the centre of gravity of the generating domain*, the naturally twisted bars are classified as:
- Naturally twisted straight bars - the axis of revolution intersects the generating domain in its center of gravity. In this case, the curve generated by the centers of gravity of the cross sections is a straight curve identical to the axis of the helical movement of the generating domain.
- Naturally twisted curved bars - the axis of revolution intersects the generating domain in a different point from its center of gravity. In this case, the curve generated by the centers of gravity of the cross sections is a helical curve.

Depending on *the nature of the helical no degenerate curves*, the naturally twisted bars are classified as follows:
- Cylindrical naturally twisted bars - the no degenerate helical curves are helical cylindrical curves. In this case, the cross section dimensions are preserved along the length of the bar.
- Non cylindrical naturally twisted bars - the no degenerate helical curves are helical non cylindrical curves (e.g. helical conical curves). In this case, the cross section is continuously variable along the length of the bar, and any two cross sections are homothetic.

Considering the pitch of the no degenerate helical curves, the naturally twisted bars can be with constant or variable pitch.

2. Assumptions and general formulas

It is considered a cylindrical naturally twisted bar, with constant pitch. The cross section of the bar has a general shape and the reference axes are the central principal axes of inertia. The longitudinal axis of the bar is the third axis of the reference system. The origin of the reference system is the centre of gravity of an end cross-section of the bar and the positive sense of this axis is toward the other end cross-section of the bar.

The bar is subjected to bending. Along the bar, the bending moment is applied in any cross section of the bar in the centre of gravity of the section and with the same direction.

The material is homogeneous and isotropic and has linear-elastic behavior. It is assumed that at any point of the bar the induced stresses do not exceed the limit of proportionality of the material.

Also it is accepted the validity of the small deformations hypothesis.

In Fig. 1 are presented two cross sections of the bar: the section located in the origin of the reference system and a section of abscissa x.

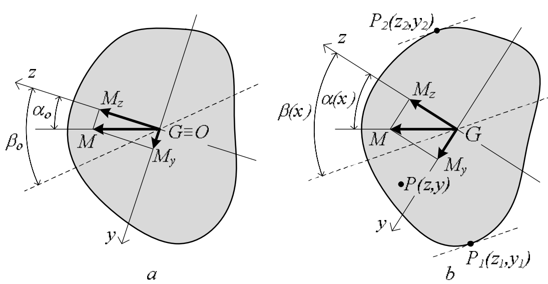

Fig. 1 – Cross sections of the twisted bar
a – Cross section located in the origin of the reference system; b – Cross section of abscissa x

The notations that were used in Fig. 1 have the following meanings:

Gz, Gy - the central principal axes of inertia of the cross section;

$\alpha_0, \alpha(x)$ - the angle between the direction of the bending moment and Gz axis, in cross section from origin, respectively in cross section with abscissa x;

$\beta_0, \beta(x)$ - the angle between the neutral axis and Gz axis, in cross section from origin, respectively in cross section with abscissa x;

M, M_z, M_y - the bending moment and its components on the central principal axes of the cross section.

Let be I_z and I_y second moments of inertia of the cross section bar with respect to the central principal axes of inertia.

The studied bar being a cylindrical naturally twisted bar with constant pitch, between the length of the bar and the pitch of the no degenerate helical curves it occurs, the following relationship:

$$p = \frac{l}{n} \tag{1}$$

where $n \in N^*$ is the number of turns of the generating domain.

Also, between angles α_0 and $\alpha(x)$ there is obviously the following relationship:

$$\alpha(x) = \alpha_0 + \frac{2\pi x}{p} = \alpha_0 + \frac{2\pi n x}{l}, \quad x \in [0, l] \tag{2}$$

Considering a point $P(z, y)$ in cross section with abscissa x, the bending normal stress in this point is given by:

$$\sigma(x) = \frac{y M_z(x)}{I_z} - \frac{z M_y(x)}{I_y} \tag{3}$$

where the components M_z and M_y of the bending moment M are:

$$\begin{cases} M_z(x) = M(x)\cos\left(\alpha_0 + \dfrac{2\pi n x}{l}\right) \\ M_y(x) = M(x)\sin\left(\alpha_0 + \dfrac{2\pi n x}{l}\right) \end{cases}, \quad x \in [0, l] \tag{4}$$

The slope of the neutral axis (see Figure 1) is:

$$\tan(\beta(x)) = \frac{I_z}{I_y}\tan\left(\alpha_0 + \frac{2\pi n x}{l}\right), \quad x \in [0, l] \tag{5}$$

The extreme values of the bending stress in cross section with abscissa x occur in points P_1 and P_2 which are at the maximum distance from the neutral axis (see fig. 1b):

$$\begin{cases} \sigma_{max}(x) = \dfrac{y_1 M_z(x)}{I_z} + \dfrac{z_1 M_y(x)}{I_y} \\ \sigma_{min}(x) = -\dfrac{y_2 M_z(x)}{I_z} - \dfrac{z_2 M_y(x)}{I_y} \end{cases}, \quad x \in [0, l] \tag{6}$$

For deflections calculus, we note that under the action of the bending moment, in cross section with abscissa x, its centre of gravity is moving on a perpendicular direction on the neutral axis.

In other words, if $u = u(x)$ and $v = v(x)$ are the components of the displacement $f = f(x)$ of the centre of gravity (see Fig. 2), we can write the following relationships:

$$f(x) = \sqrt{u^2(x) + v^2(x)},\ x \in [0, l] \quad (7)$$

$$\tan(\theta(x)) = \tan(\beta(x)),\ x \in [0, l] \quad (8)$$

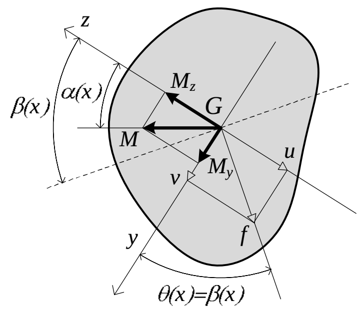

Fig. 2 – Displacement of the centre of gravity in cross section with x abscissa

The displacements $u = u(x)$ and $v = v(x)$ can be calculated by integrating well-known approximate differential equations:

$$\begin{cases} \dfrac{d^2u}{dx^2} \cong -\dfrac{M_y(x)}{EI_y}, \\ \dfrac{d^2v}{dx^2} \cong -\dfrac{M_z(x)}{EI_z} \end{cases},\ x \in [0, l] \quad (9)$$

For the bi-univocal determination of the elastic line of the bar it is also necessary to find the angle $\varphi = \varphi(x)$ of rotation in current section. This angle can be calculated with:

$$\varphi(x) = \sqrt{\varphi_z^2(x) + \varphi_y^2(x)},\ x \in [0, l] \quad (10)$$

Where the rotational components are given by the following differential equations:

$$\varphi_z(x) \cong \frac{dv}{dx},\ \varphi_y(x) \cong \frac{du}{dx},\ x \in [0, l] \quad (11)$$

At the end of this paragraph, we note that the calculus particularities in case of the naturally straight twisted bars bending are dictated mainly by the fact that along the bar, the central principal axes of the cross section are rotating. In this context, the central *principal axes are with variable directions*. Thus, the cross section of the bar is constant in terms of geometric form (size and shape), but it has a rotated variable position with respect to a reference section (e.g. cross section from the origin of the reference system).

Another very important aspect is that the components $M_z(x)$ and $M_y(x)$ of the bending moment, are depending both on the value of the bending moment in current cross section $M(x)$ and the angle of rotation $\alpha(x)$.

Therefore the section where is developed the greatest maximum bending stress is not necessarily the cross section where the maximum bending moment occurs.

3. Case study

In this paragraph is presented a case study to highlight the peculiarities of calculus and way of solving it.

The objective of the case study is to highlight issues related to locate the position of the dangerous section and to calculate the greatest maximum bending stress.

In this context, is considered a cylindrical naturally twisted cantilever bar with constant pitch. The twisted bar has a rectangular cross section and in they rigid fixing end (which is considered the origin of the longitudinal axis), is verifies the relation $\alpha_0 = 0$ (see Fig. 3).

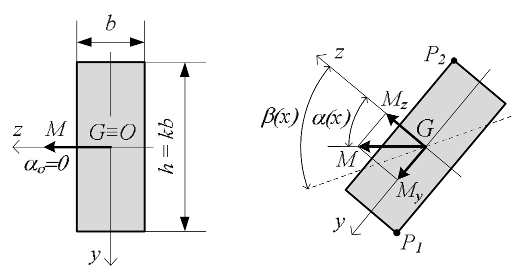

Fig. 3 – The cross sections of the cylindrical naturally twisted cantilever bar

In Fig. 3 are presented a cross section from origin of the reference system and the cross section with abscissa x. With b and $h = kb$ ($k = ct.$, $k \in R_+$, $(\forall) x \in [0, l]$) are noted the width and the height of the cross section.

Since in rectangular section, the most distant points from the neutral axis are also the most distant points from the central principal axes of inertia, the maximum bending stresses in the current cross section are given by the following relations:

$$\begin{cases} \sigma_{max}(x) = \dfrac{|M_z(x)|}{W_z} + \dfrac{|M_y(x)|}{W_y} > 0 \\ \sigma_{min}(x) = -\sigma_{max}(x) \end{cases},\ x \in [0, l] \quad (12)$$

Where:

$$W_z = \frac{1}{6}k^2b^3,\ W_y = \frac{1}{6}kb^3 \qquad (13)$$

Given the general relationship (4), in relationship (12) we have:

$$M_z(x) = |M(x)| \cdot \left|\cos\left(\frac{2\pi nx}{l}\right)\right|,\ x \in [0,l] \qquad (14)$$

respectively

$$M_y(x) = |M(x)| \cdot \left|\sin\left(\frac{2\pi nx}{l}\right)\right|,\ x \in [0,l] \qquad (15)$$

The cantilever bar is loaded by a concentrated moment applied to the free end, as seen in Figure 4.

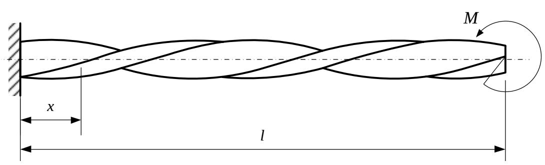

Fig. 4 – Cantilever naturally twisted bar loaded by a concentrated moment applied to the free end

Because the bending moment is constant along the bar ($|M(x)| = M = ct.,\ x \in [0,l]$), in cross section with x abscissa, the maximum bending stress is given by:

$$\sigma_{max}(x) = \frac{6M}{k^2b^3}\left(\left|\cos\left(\frac{2\pi nx}{l}\right)\right| + k\left|\sin\left(\frac{2\pi nx}{l}\right)\right|\right), \qquad (16)$$
$$x \in [0,l]$$

The position of the dangerous section and the corresponding maximum bending stress (the greatest maximum bending stress along the bar) involves solving the equation:

$$\frac{d}{dx}(\sigma_{max}(x)) = 0 \qquad (17)$$

which is equivalent to solve the equation:

$$\frac{d}{dx}\left(\left|\cos\left(\frac{2\pi nx}{l}\right)\right| + k\left|\sin\left(\frac{2\pi nx}{l}\right)\right|\right) = 0 \qquad (18)$$

Equation (19) can not be solved until after explaining the function module. Noting that the argument of the trigonometric functions (19), in different cross section, is in different quadrants, depending on abscissa of the cross section, after calculation we obtain the following equations:

$$\begin{cases} -\sin(2n\pi x/l) + k\cos(2n\pi x/l) = 0,\ x \in J_1 \\ \sin(2n\pi x/l) + k\cos(2n\pi x/l) = 0,\ x \in J_2 \\ \sin(2n\pi x/l) - k\cos(2n\pi x/l) = 0,\ x \in J_3 \\ -\sin(2n\pi x/l) - k\cos(2n\pi x/l) = 0,\ x \in J_4 \end{cases} \qquad (19)$$

with

$$\begin{cases} J_1 = [ql/n;(4q+1)l/(4n)) \\ J_2 = [(4q+1)l/(4n);(4q+2)l/(4n)) \\ J_3 = [(4q+2)l/(4n);(4q+3)l/(4n)) \\ J_4 = [(4q+3)l/(4n);(q+1)l/n) \end{cases} \qquad (20)$$

where $q \in Z_+$.

The valid solution in the range $[0;2\pi]$ for any of the equations (19) can be expressed by:

$$x_v \in \left\{\pm\frac{l}{2n\pi}\arctan k \cap J_i\right\} \qquad (20)$$

where $J_i \equiv J_1$, or $J_i \equiv J_2$, or $J_i \equiv J_3$, or $J_i \equiv J_4$, depending on which equation (19) is solved. The dangerous cross section is the cross section where the bending stress is

$$\sigma_{max\,max} = \max\{\sigma_{max}(x_v)\} \qquad (21)$$

4. Conclusions

The calculation of bending stresses and deformations for the naturally twisted bars is difficult even in the cases that are characterized by significant simplifications. The numerical approach by finite element modeling is a perfectly viable alternative, although the obtaining of the optimal mesh for such geometry is not exactly simple. However, it is important to note that the major advantage of the theoretical study (when it is possible) is given by the possibility of generalizing the results, and finding the optimum geometry through the study of mathematical functions and not by successive attempts, as the numerical approach.

The case study considered a loading for which the bending moment is expressed through a unique function along the length of the bar. Consequently, the intervals for which the general equation (17) has been solved were determined from the condition that the module functions (18) were represented by a unique expression. Similarly there can be treated other cases in which the loads induce a bending moment which can be expressed through a unique function along the bar.

References

[1] **Dumitrache P.** – *Rezistenţa materialelor în aplicaţii*, digital format, Faculty of Engineering Brăila, 2007.