ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
MATHEMATICS, PHYSICS, THEORETICAL MECHANICS
FASCICLE II, YEAR X (XLI) 2018, No. 2

Article DOI: https://doi.org/10.35219/ann-ugal-math-phys-mec.2018.2.03

NEW SEQUENCE AND INEQUALITIES ASSOCIATED WITH THE EULER-MASCHERONI CONSTANT USING THE SUM OF INVERSE ODD NATURAL NUMBERS

Jenică Crînganu

"Dunărea de Jos" University of Galati, Faculty of Science and Environment, Department of Mathematics and Computer Science, 111 Domneasca street, Galati, Romania
e-mail: jcringanu@ugal.ro

Abstract
Using the classical sequence that converges to the Euler-Mascheroni constant, we will define a new sequence and new inequalities associated with the Euler-Mascheroni constant using the sum of inverses numbers of odd natural numbers. As a consequence we establish an estimate for the sum of inverses of odd natural numbers.

1. INTRODUCTION

It is well known that the sequence

$$\gamma_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n, \quad n \geq 1,$$

is convergent to a limit denoted γ = 0,5772... now known as Euler-Mascheroni constant. Many authors have obtained different estimates for $\gamma_n - \gamma$, for exemple the following increasingly better

Many estimations have been given in the literature for $\gamma_n - \gamma$. We recall some of them:

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2(n-1)}, \; n \geq 2, \qquad [8]$$

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}, \; n \geq 1, \qquad [10]$$

$$\frac{1-\gamma}{n} < \gamma_n - \gamma < \frac{1}{2n}, \; n \geq 1, \qquad [2]$$

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \; n \geq 1, \qquad [6, 7]$$

$$\frac{1}{2n+\frac{2}{5}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, \; n \geq 1, \qquad [9]$$

$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} \leq \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, \; n \geq 1, \qquad [1, 9]$$

A simple calculus (see [3]) shows that for all $a > \frac{1}{3}$ there exists $n_a \in N$ such that

$$\frac{1}{2n+a} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, \text{ for all } n \geq n_a.$$

If $x_n = 1 + \frac{1}{3} + ... + \frac{1}{2n-1}$ and

$$y_n = \frac{1}{2} + \frac{1}{4} ... + \frac{1}{2n} = \frac{1}{2}(1 + \frac{1}{2} + ... + \frac{1}{n}) = \frac{1}{2}(\gamma_n - \gamma) + \frac{1}{2}(\ln n + \gamma),$$

then by the above inequalities it results that for all $a > \frac{1}{3}$ there exists $n_a \in N$ such that

$$\frac{1}{2(2n+a)} < y_n - \frac{1}{2}\ln n - \frac{1}{2}\gamma < \frac{1}{2(2n+\frac{1}{3})}, \text{ for all } n \geq n_a.$$

The convergence of the sequence $(y_n - \frac{1}{2}\ln n)$ to $\frac{1}{2}\gamma$ is very slow. With a modified sequence (x_n) we obtain a faster convergences and we prove that for all $a > 0$ there exists $n_a \in N$ such that

$$\frac{1}{48(n+a)^2} < x_n - \frac{1}{2}\ln(4n) - \frac{1}{2}\gamma < \frac{1}{48n^2}, \text{ for all } n \geq n_a.$$

From the definition of γ_n we have $\gamma_{2n} = x_n + y_n - \ln(2n) = x_n - \frac{1}{2}\ln(4n) + \frac{1}{2}\gamma_n$.

By the convergence of γ_n to γ it results that the sequence $a_n = x_n - \frac{1}{2}\ln(4n)$ converge to $\frac{1}{2}\gamma$.

Now we define the sequence $b_n = x_n - \frac{1}{2}\ln(4n) - \frac{1}{2}\gamma$. The tool for measuring the speed of convergence is a result stated by Mortici [5] according to which a sequence b_n converging to zero is the fastest possible when the difference $b_n - b_{n+1}$ is the fastest possible. More precisely, if there exists the $\lim_{n\to\infty} n^k(b_n - b_{n+1}) = l$, then $\lim_{n\to\infty} n^{k-1}b_n = \frac{l}{k-1}$.

Recent results using this lemma were obtained for example in [2, 4-6].

In our case of b_n, we have $b_n - b_{n+1} = \frac{1}{2}\ln(1+\frac{1}{n}) - \frac{1}{2n+1}$, and using a Mac-Laurin growth serie we get $b_n - b_{n+1} = \frac{1}{24n^3} + O(\frac{1}{n^4})$, and so $\lim_{n\to\infty} n^3(b_n - b_{n+1}) = \frac{1}{24}$.

By the above result we obtain $\lim_{n\to\infty} n^2 b_n = \frac{l}{k-1} = \frac{1}{48}$.

2. THE MAIN RESULT

Theorem 2. 1. (i) For every $n \geq 1$ we have

$$a_n - \frac{1}{2}\gamma < \frac{1}{48n^2};$$

(ii) For every $a > 0$ there exists $n_a \in N$ such that

$$\frac{1}{48(n+a)^2} < a_n - \frac{1}{2}\gamma \quad \text{for all } n \geq n_a.$$

Proof. We define the sequence

$$c_n = a_n - \frac{1}{2}\gamma - \frac{1}{48(n+a)^2} = 1 + \frac{1}{3} + ... + \frac{1}{2n-1} - \frac{1}{2}\ln(4n) - \frac{1}{2}\gamma - \frac{1}{48(n+a)^2}, \text{ for } a \geq 0,$$

and so $c_{n+1} - c_n = f(n),$ where

$$f(n) = \frac{1}{2n+1} - \frac{1}{2}\ln(4n+4) + \frac{1}{2}\ln(4n) - \frac{1}{48(n+a+1)^2} + \frac{1}{48(n+a)^2}.$$

The derivative of function f is equal to

$$f'(n) = -\frac{2}{(2n+1)^2} - \frac{1}{2(n+1)} + \frac{1}{2n} + \frac{1}{24(n+a+1)^3} - \frac{1}{24(n+a)^3} =$$

$$= \frac{P(n)}{24n(n+1)(2n+1)^2(n+a)^3(n+a+1)^3},$$

where

$$P(n) = 48an^5 + (168a^2 + 120a - 7)n^4 + 2(120a^3 + 168a^2 + 45a - 7)n^3 +$$
$$+ (180a^4 + 360a^3 + 201a^2 + 15a - 8)n^2 + (72a^5 + 180a^4 + 144a^3 + 33a^2 - 3a - 1)n + 12a^3(a+1)^3.$$

(i) If $a = 0$ then $P(n) = -7n^4 - 14n^3 - 8n^2 - n < 0,$
for all $n \geq 1$ and then f is strictly decreasing. Since $f(\infty) = 0$ it follows that $f(n) > 0$ for all $n \geq 1,$ so that (c_n) is strictly increasing.

Since (c_n) converges to zero it follows that $c_n < 0$ for all $n \geq 1,$ so that

$$a_n - \frac{1}{2}\gamma < \frac{1}{48n^2}, \quad \text{for all } n \geq 1.$$

(ii) If $a > 0$ then there exists $n_a \in N$ such that $P(n) > 0$ for all $n \geq n_a$ and then f is strictly increasing on $[n_a, \infty).$ Since $f(\infty) = 0$ it results that $f(n) < 0$ for all $n \geq n_a,$ so that $(c_n)_{n \geq n_a}$ is strictly decreasing.

Since (c_n) converges to zero it follows that $c_n > 0$ for all $n \geq n_a,$ so that

$$\frac{1}{48(n+a)^2} < a_n - \frac{1}{2}\gamma \quad \text{for all } n \geq n_a. \qquad \blacksquare$$

Remark. By this theorem it results that for every a $a > 0$ there exists $n_a \in N$ such that

$$\frac{1}{48(n+a)^2} + \frac{1}{2}\ln(4n) + \frac{1}{2}\gamma < 1 + \frac{1}{3} + ... + \frac{1}{2n-1} < \frac{1}{48n^2} + \frac{1}{2}\ln(4n) + \frac{1}{2}\gamma, \text{ for all } n \geq n_a.$$

Now we find the constant n_a in some particular case.

For exemple, if $a = 0{,}1 = \frac{1}{10},$ then

$$P(n) = \frac{24}{5}n^5 + \frac{167}{25}n^4 + \frac{7}{5}n^3 - \frac{514}{125}n^2 - \frac{10091}{12500}n - \frac{3993}{250000} > 0,$$

for all $n \geq 1,$ and so

$$\frac{1}{48(n+\frac{1}{10})^2}+\frac{1}{2}\ln(4n)+\frac{1}{2}\gamma<1+\frac{1}{3}+...+\frac{1}{2n-1}<\frac{1}{48n^2}+\frac{1}{2}\ln(4n)+\frac{1}{2}\gamma, \text{ for all } n\geq 1.$$

If $a=0{,}01=\frac{1}{100}$, then

$$P(n)=\frac{12}{25}n^5-\frac{7229}{1250}n^4-\frac{163327}{12500}n^3-\frac{39147691}{5000000}n^2-\frac{1283192741}{1250000000}n-\frac{3090903}{250000000000}>0,$$

for all $n\geq 16$, and so

$$\frac{1}{48(n+\frac{1}{100})^2}+\frac{1}{2}\ln(4n)+\frac{1}{2}\gamma<1+\frac{1}{3}+...+\frac{1}{2n-1}<\frac{1}{48n^2}+\frac{1}{2}\ln(4n)+\frac{1}{2}\gamma, \text{ for all } n\geq 16.$$

Let us remark that a direct calculus shows that these inequalities hold and for $n\in\{9,10,11,12,13,14,15\}$ and then

$$\frac{1}{48(n+\frac{1}{100})^2}+\frac{1}{2}\ln(4n)+\frac{1}{2}\gamma<1+\frac{1}{3}+...+\frac{1}{2n-1}<\frac{1}{48n^2}+\frac{1}{2}\ln(4n)+\frac{1}{2}\gamma, \text{ for all } n\geq 9.$$

References

[1] H. Alzer, *Inequalities for the gamma and polygamma functions*, Abh. Math. Sem. Univ. Hamburg 68 (1998) 363-372;

[2] G.D. Anderson, R.W. Barnard, K.C. Richards, M.K. Vamanamurthy, M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. 345 (1995) 1713-1723;

[3] J. Cringanu, On *the classical convergent sequence to the Euler-Mascheroni constant*, Annals of “Dunarea de Jos" University of Galati, Mathematics, Physics, Theoretical Mechanics, Fascicle II, Year VI, (XXXVII)(2015), 247-249;

[4] D.W. DeTemple, *A quicker convergence to Euler's constant*, Amer. Math. Monthly 100 (5) (1993) 468-470;

[5] C. Mortici, *On new sequences converging towards the Euler-Mascheroni constant*, Computers and Mathematics with Applications 59 (2010) 2610-2614;

[6] C. Mortici, A. Vernescu, *An improvement of the convergence speed of the sequence* (γ_n) *converging to Euler's constant*, An. Stiint. Univ. "Ovidius" Constanta 13 (1) (2005) 97-100;

[7] C. Mortici, A. Vernescu, *Some new facts in discrete asymptotic analysis*, Math. Balkanica (NS) 21 (Fasc.3-4) (2007) 301-308;

[8] S. R. Tims, J. A. Tyrrel, *Approximate evaluation of Euler's constant*, Math. Gaz., 55 (1971) 65-67;

[9] L. Toth, *Probem E3432*, Amer. Math. Monthly, 98 (3) (1991) 264;

[10] R. M. Young, *Euler's constant*, Math. Gaz. 75 (472) (1991) 187-190.