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EXTENSION OF NOETHER'S THEOREM FOR THE RHEONOMIC LAGRANGIAN OF SECOND ORDER

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Abstract
This paper extends the Noether's theorem for the rheonomic Lagrangian of second order.

Keywords: rheonomic, Lagrange space, simmetry, action integral

1. INTRODUCTION

Time dependent Lagrangians are involved in many problems in Physics and Variational calculus. An explicit dependence on time in problems of dynamics has been considered by A. Wundheiler in paper [4] and he called the geometrical background related to it "Rheonomic Geometry". An extension of Lagrange geometry, i.e. that of the rheonomic Lagrange spaces was studied by M. Anastasiei in [1].

Noether's theorem is a property of any system that can be derived from an action and possesses some continuous symmetry. In words, to any given symmetry, Noether's algorithm associates a conserved charge to it.

In this paper we'll study different symmetries of a rheonomic Lagrange space of second order and their relations with conservation laws.

2. MAIN RESULTS

Let M be a C^∞ manifold of dimension n and the bundle (Osc^2M,π, M). [3]
A point $x \in M$ has the local coordinates denoted with (x^i), $i=\overline{1,n}$ and the local coordinates of $u \in Osc^2M$ are $(x^i, y^{(1)i}, y^{(2)i})$. The bundle Osc^2M can be identified with the bundle of accelerations of second order of M, denoted with T^2M. [3]

The differentiable bundle

$$E= (T^2M\times\boldsymbol{R},\pi, M)$$

is determined by the differential manifold $T^2M\times\boldsymbol{R}$ and the canonical projection $\pi: T^2M\times\boldsymbol{R}\to M$

$$\pi(x,y^{(1)}, y^{(2)},t)=x, \qquad \forall (x,y^{(1)}, y^{(2)},t) \in T^2M\times\boldsymbol{R}.$$

A rheonomic Lagrangian of second order is a mapping $L: E= T^2M\times\boldsymbol{R} \to\boldsymbol{R}$

$$(x, y^{(1)}, y^{(2)}, t) \to L(x, y^{(1)}, y^{(2)}, t) \in\boldsymbol{R}.$$

The Lagrangian L is called *differentiable* if it is of class C^∞ on $E=T^2M\times\boldsymbol{R}$ and continuous in the points $(x,0,0,t)\in T^2M\times\boldsymbol{R}$..

A symmetry for the rheonomic differential Lagrangian $L(x,y^{(1)},y^{(2)},t)$ is a C^{∞} diffeomorphism $\varphi:M\times\mathbf{R}\to M\times\mathbf{R}$, which preserves the variational principle for the action integral

$$I(c)=\int_0^1 L\left(x,\frac{dx}{dt},\frac{1}{2}\frac{d^2x}{dt^2},t\right)dt\,. \tag{1}$$

We will consider a local symmetry for the rheonomic Lagrangian of second order L, using the local diffeomorphism φ defined on an open subset $U\times(a,b)\subset M\times\mathbf{R}$.

Now we study the infinitesimal symmetries on the open subset $U\times(a,b)\subset M\times\mathbf{R}$, defined by

$$x'^i = x^i + \varepsilon V^i(x,t) \quad (i=1,\dots,n) \tag{2}$$
$$t' = t + \varepsilon\,\tau(x,t)$$

where ε is a real number, sufficiently small in absolute value so that the points (x,t) and (x',t') belong to the same domain of a local map $U\times(a,b)$ where the parametrized curve $c:t\in[0,1]\to(x^i(t),t)\in U\times(a,b)$ is defined. $V^i(x,t)$ is a vectorial field on the open set $U\times(a,b)$.

The inverse of the local diffeomorphism (2) is given by:

$$x^i = x'^i - \varepsilon V^i(x,t) \tag{3}$$
$$t = t' - \varepsilon\,\tau(x,t).$$

Vectorial field $V^i(x(t),t)=V^i(t)$ satisfies in the endpoints $c(0)$ and $c(1)$ the following equations

$$V^i(0)=V^i(1)=0;\quad \frac{dV^i}{dt}(0)=\frac{dV^i}{dt}(1)=0\,.$$

An infinitesimal transformation (2) is a symmetry for Lagrangian $L(x,y^{(1)},y^{(2)},t)$ if and only if for any function C^{∞} differentiable. $\Phi(x,y^{(1)},t)$ the following equality holds

$$L\left(x'(t),\frac{dx'}{dt'},\frac{1}{2}\frac{d^2x'}{dt'^2},t'\right)dt'=\left\{L\left(x,\frac{dx}{dt},\frac{1}{2}\frac{d^2x}{dt^2},t\right)+\Phi\left(x,\frac{dx}{dt},t\right)\right\}dt\,. \tag{4}$$

From (2) one obtains

$$\frac{dt'}{dt}=1+\varepsilon\frac{d\tau}{dt};\quad \frac{dx'^i}{dt'}=\frac{dx^i}{dt}+\varepsilon\varphi^{(1)i};\quad \frac{1}{2}\frac{d^2x'^i}{dt'^2}=\frac{1}{2}\left[\frac{d^2x^i}{dt^2}+\varepsilon\,\varphi^{(2)i}\right] \tag{5}$$

where

$$\varphi^{(1)i}=\frac{dV^i}{dt}-\frac{dx^i}{dt}\frac{d\tau}{dt};\qquad \varphi^{(2)i}=\frac{d^2V^i}{dt^2}-2\frac{d^2x^i}{dt^2}\frac{d\tau}{dt}-\frac{dx^i}{dt}\frac{d^2\tau}{dt^2}\,. \tag{6}$$

If we substitute (5) and (6) into (4) and we neglect the terms in ε, ε², … , we replace Φ with εΦ, one obtains

$$\frac{\partial L}{\partial t}\tau+\frac{\partial L}{\partial x^i}V^i+\frac{\partial L}{\partial y^{(1)i}}\varphi^{(1)i}+\frac{1}{2}\frac{\partial L}{\partial y^{(2)i}}\varphi^{(2)i}+L\frac{d\tau}{dt}=\frac{d\Phi}{dt}\,. \tag{7}$$

Using scalar fields

$$\overset{1}{I}(L)=y^{(1)i}\frac{\partial L}{\partial y^{(2)i}};\qquad \overset{2}{I}(L)=y^{(1)i}\frac{\partial L}{\partial y^{(1)i}}+2y^{(2)i}\frac{\partial L}{\partial y^{(2)i}}\,.$$

equation (7) can be written under the following form

$$V^i\frac{\partial L}{\partial x^i}+\frac{dV^i}{dt}\frac{\partial L}{\partial y^{(1)i}}+\frac{1}{2}\frac{d^2V^i}{dt^2}\frac{\partial L}{\partial y^{(2)i}}+\left\{L\frac{d\tau}{dt}+\tau\frac{\partial L}{\partial t}-\left[\overset{2}{I}(L)\frac{d\tau}{dt}+\frac{1}{2}\overset{1}{I}(L)\frac{d^2\tau}{dt^2}\right]\right\}=\frac{d\Phi}{dt}\,. \tag{8}$$

Using the operator $\frac{d_V}{dt} = V^i \frac{\partial}{\partial x^i} + \frac{dV^i}{dt}\frac{\partial}{\partial y^{(1)i}} + \frac{1}{2}\frac{d^2V^i}{dt^2}\frac{\partial}{\partial y^{(2)i}}$ we have the theorem:

Theorem 1. A necessary and sufficient condition for infinitesimal transformation (2) to be a symmetry for the rheonomic Lagrangian *L(x, y⁽¹⁾,y⁽²⁾,t)* is the left-hand side of equality

$$\frac{d_v L}{dt} + \left\{ L\frac{d\tau}{dt} + \tau\frac{\partial L}{\partial t} - \left[\overset{2}{I}(L)\frac{d\tau}{dt} + \frac{1}{2}\overset{1}{I}(L)\frac{d^2\tau}{dt^2} \right] \right\} = \frac{d\Phi}{dt} \qquad (9)$$

be equal to $\frac{d}{dt}\Phi(x,y^{(1)},t)$ on the smooth curve *c*.

A straightforward calculation leads to following theorem.

Theorem 2. Along any smooth curve *c* from the manifold *M*, for any rheonomic Lagrangian of second order the following relation holds:

$$\frac{d_V L}{dt} = \overset{o}{E}_i(L)V^i + \frac{d}{dt}\left(\overset{2}{I}_V(L)\right) - \frac{1}{2}\frac{d^2}{dt^2}\left(\overset{1}{I}_V(L)\right),$$

where $\overset{o}{E}_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y^{(2)i}}\right)$.

Along the curve c, the function defined by

$$\overset{1}{\varepsilon}_c(L) = -\frac{1}{2}\overset{1}{I}(L)$$

is called the energy of first order for the Lagrangian $L(x,y^{(1)},y^{(2)},t)$,

and the function defined by

$$\overset{2}{\varepsilon}_c(L) = \overset{2}{I}(L) - \frac{1}{2}\frac{d}{dt}\overset{1}{I}(L) - L(x, y^{(1)},y^{(2)},t)$$

is named the energy of second order for Lagrangian $L(x,y^{(1)},y^{(2)},t)$.

Theorem 3. For every smooth curve c on the base manifold *M* the following formula holds

$$\frac{d\overset{1}{\varepsilon}(L)}{dt} + \frac{1}{2}\overset{2}{I}(L) = -\frac{1}{2}\frac{dx^i}{dt}\overset{1}{E}_i(L),$$

with $\overset{1}{E}_i = -\frac{\partial}{\partial y^{(1)i}} + \frac{d}{dt}\frac{\partial}{\partial y^{(2)i}}$.

According Theorem 2 and Theorem 3, equation (9) can be written in the following form

$$V^i\overset{o}{E}_i(L) + \tau\frac{\partial L}{\partial t} + \frac{d}{dt}\overset{2}{I}_V(L) - \frac{1}{2}\frac{d^2}{dt^2}\overset{1}{I}_V(L) + \tau\frac{d}{dt}\overset{2}{\varepsilon}_c(L) + \frac{d}{dt}\left[-\tau\overset{2}{\varepsilon}_c(L) + \frac{d\tau}{dt}\overset{1}{\varepsilon}_c(L)\right] = \frac{d\Phi}{dt} \Leftrightarrow$$

$$V^i\overset{o}{E}_i(L) + \tau\left(\frac{\partial L}{\partial t} + \frac{d\overset{2}{\varepsilon}_c}{dt}\right) + \frac{d}{dt}\left(\overset{2}{I}_V(L) - \frac{1}{2}\frac{d}{dt}\overset{1}{I}_V(L) - \tau\overset{2}{\varepsilon}_c(L) + \overset{1}{\varepsilon}_c(L)\frac{d\tau}{dt} - \Phi\right) = 0.$$

Using the above equation we can write a theorem of type Noether

Theorem 4. For any infinitesimal symmetry (2) (which verify (9)) of a rheonomic Lagrangian $L(x,y^{(1)},y^{(2)},t)$ and for any C^∞ function, $\Phi(x, y^{(1)},t)$, the following function

$$F(L,\Phi)= \overset{2}{I}_{v}(L) - \frac{1}{2}\frac{d}{dt}\overset{1}{I}_{v}(L) - \tau\, \overset{2}{\varepsilon}_{c}(L) + \frac{d\tau}{dt}\overset{1}{\varepsilon}_{c}(L) - \Phi \qquad (10)$$

is conserved on the solution curves of the Euler-Lagrange equations

$$\overset{o}{E}_{i}(L)=0,\ y^{(1)i}=\frac{dx^i}{dt},\ y^{(2)i}=\frac{1}{2}\frac{d^2x^i}{dt^2}.$$

Especially if the Zermelo conditions are checked this results lead us to obtain a new theorem of type Noether.

Theorem 5. For any infinitesimal symmetry (2) of a rheonomic Lagrangian $L(x,y^{(1)},y^{(2)},t)$, which verify the Zermelo equations

$$\overset{1}{I}_{v}(L)=0,\ \overset{2}{I}_{v}(L)=L$$

and for any C^{∞} function $\Phi(x,y^{(1)},t)$, the following function

$$F(L,\Phi)= \overset{2}{I}_{v}(L) - \frac{1}{2}\frac{d}{dt}\overset{1}{I}_{v}(L) - \Phi \qquad (11)$$

is conserved along the solution curves of the Euler-Lagrange equations $\overset{o}{E}_{i}(L)=0$, $y^{(1)i}=\frac{dx^i}{dt}$, $y^{(2)i}=\frac{1}{2}\frac{d^2x^i}{dt^2}$.

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