ANNALS OF “DUNAREA DE JOS” UNIVERSITY OF GALATI
MATHEMATICS, PHYSICS, THEORETICAL MECHANICS
FASCICLE II, YEAR XIV (XLV) 2022, No. 2
DOI: https://doi.org/10.35219/ann-ugal-math-phys-mec.2022.2.02

On the Hilbert function of vertex cover algebras of Cohen-Macaulay bipartite graphs

Cristian Ion

“Dunarea de Jos” University of Galati, Faculty of Sciences and Environment,
111 Domneasca street, 800201 Galati, Romania
Corresponding author: cristian.adrian.ion@gmail.com

Abstract
We study the $h-$vector and the Hilbert function of the vertex cover algebra $A(G)$, introduced and first studied by J. Herzog, T. Hibi and N. V. Trung ([6]), for a special class of bipartite graphs, namely for Cohen-Macaulay bipartite graphs.
Keywords: vertex cover algebra, Cohen-Macaulay bipartite graph, simplicial complex, $h-$vector, Hilbert function.

1. INTRODUCTION

In the first part of the paper, we introduce the definitions and the concepts that we operate with and we fix the notation exactly as we did it in [3]. Let $G=(V,E)$ be a simple (i.e. finite, undirected, loop less and without multiple edges) graph with vertex set $V=[n]$ and the edge set $E=E(G)$. A *vertex cover* of G is a subset $C \subset V$ such that $C \cap \{i,j\} \neq \emptyset$, for any edge $\{i,j\} \in E(G)$. A vertex cover C of G is called *minimal* if no proper subset $C' \subset C$ is a vertex cover of G. A graph G is called *unmixed* if all minimal vertex covers of G have the same cardinality. Let $R=K[x_1,x_2,...,x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* of G is the monomial ideal $I(G)$ of R generated by all the quadratic monomials $x_i x_j$ with $\{i,j\} \in E(G)$. It is said that a graph G is *Cohen-Macaulay* (over K) if the quotient ring $R/I(G)$ is Cohen-Macaulay. Every Cohen-Macaulay graph is unmixed.

A vertex cover $C \subset [n]$ can be represented as a $(0,1)$-vector c that satisfies the restriction $c(i)+c(j) \geq 1$, for every $\{i,j\} \in E(G)$. For each $k \in N$, a *vertex cover* of G *of order* k, or simply k-vertex cover of G is a vector $c \in N^n$ such that $c(i)+c(j) \geq k$, for every $\{i,j\} \in E(G)$. The *vertex cover algebra* $A(G)$ is defined as the subalgebra of the one variable polynomial ring $R[t]$ generated by all monomials $x_1^{c_1} x_2^{c_2} ... x_n^{c_n} t^k$, where $c=(c_1,c_2,...,c_n) \in N^n$ is a k-vertex cover of G. This algebra was introduced and first studied in [6].

Let m be the maximal graded ideal of R. The graded K-algebra $\overline{A}(G)=A(G)/mA(G)$ is called the *basic cover algebra* and it was introduced and first studied in [5, Section 3].

Our aim in this paper is to study the $h-$vector and the Hilbert function of the vertex cover algebra $A(G)$ for Cohen-Macaulay bipartite graphs.

Let $P_n = \{p_1, p_2, ..., p_n\}$ be a poset with partial order $\leq$. Let $G = G(P_n)$ be the bipartite graph on the set $V_n = W \cup W'$, where $W = \{x_1, x_2, ..., x_n\}$ and $W' = \{y_1, y_2, ..., y_n\}$, whose edge set $E(G)$ consists of all 2-element subsets $\{x_i, y_j\}$ with $p_i \leq p_j$. It is said that a bipartite graph on $V_n = W \cup W'$ *comes from a poset*, if there is a finite poset P_n on $\{p_1, p_2, ..., p_n\}$ such that $p_i \leq p_j$ implies $i \leq j$ and after relabeling of the vertices of G one has $G = G(P_n)$. Herzog and Hibi proved in [4, Theorem 3.4] that a bipartite graph G is Cohen-Macaulay if and only if G comes from a poset.

Example 1.1. Let $P_3 = \{p_1, p_2, p_3\}$ be the poset with $p_1 \leq p_2$. The Hasse diagram of P_3 is represented in the next figure:

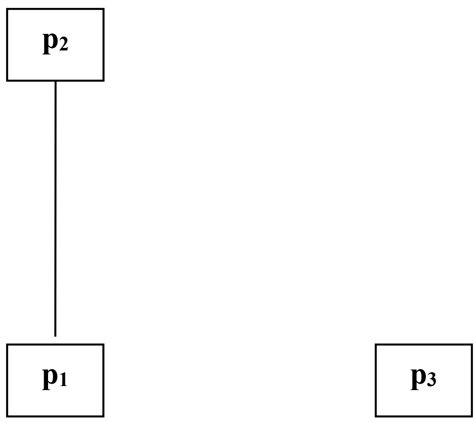

Fig. 1

The graph $G = G(P_3)$ is represented geometrically in the next figure:

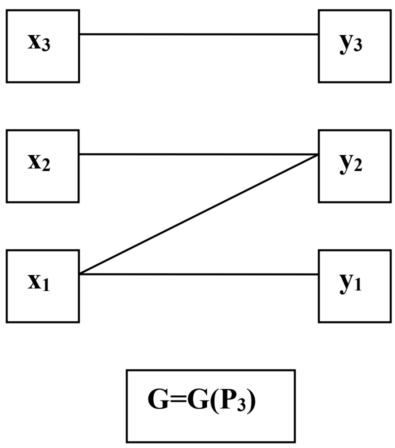

Fig. 2

In this respect, by [6, Lemma 4.1, Theorem 5.1.b], the vertex cover algebra $A(G)$ is standard graded over S and it is the Rees algebra of the vertex cover ideal I_G, which is generated by all monomials $x_1^{c_1}x_2^{c_2}...x_n^{c_n}y_1^{c_{n+1}}y_2^{c_{n+2}}...y_n^{c_{2n}}$, where the $(0.1)-$vector $c=(c_1,c_2,...,c_n,c_{n+1},c_{n+2},...,c_{2n})$ is a $1-$vertex cover of the graph G.

By [5, Lemma 2.1], there is an one-to-one correspondence between the set $M(G)$ of all minimal vertex covers of G and the lattice $I(P_n)$ of all poset ideals of P_n. Thus, it can be assigned to each minimal vertex cover of G the poset ideal $\alpha_C=\{p_i \mid x_i \in C\}$. Conversely, if α is a poset ideal of P_n, then the corresponding set $C_\alpha=\{x_i \mid p_i \in \alpha\}\cup\{y_j \mid p_j \notin \alpha\}$ is a minimal vertex cover of G.

Let P_3 be the poset from the Example 1.1. The set of all minimal vertex covers of G is:

$$M(G)=\{\{y_1,y_2,y_3\},\{x_1,y_2,y_3\},\{x_3,y_1,y_2\},\{x_1,x_2,y_3\},\{x_1,x_3,y_2\},\{x_1,x_2,x_3\}\},$$

and the lattice of all poset ideal of P_3 is:

$$I(P_3)=\{\emptyset,\{p_1\},\{p_3\},\{p_1,p_2\},\{p_1,p_3\},\{p_1,p_2,p_3\}\}.$$

The distributive lattice $(I(P_3),\subset)$ is represented graphically in the next figure:

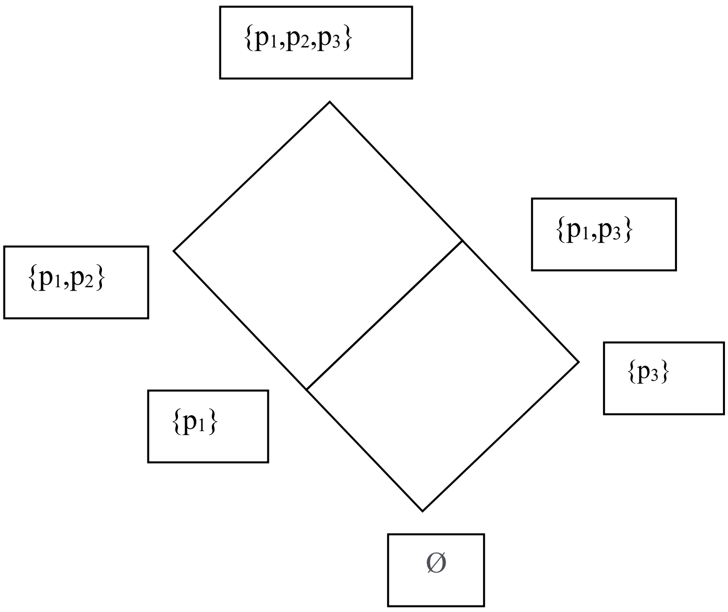

Fig. 3

In Section 2, we study the $h-$vector and the Hilbert function of $A(G)$. We assign to each poset $P_n=\{p_1,p_2,...,p_n\}$ such that $p_i \le p_j$ implies $i \le j$ a simplicial complex Δ_{P_n} on the set $[n]\cup I(P_n)$ whose Stanley-Reisner ideal coincides with the initial ideal of the toric ideal Q_G of $A(G)$ with respect to a suitable monomial order. (See [4, Section 1] for the definition and the properties of the toric ideal of $A(G)$). The simplicial complex Δ_{P_n} plays a key role in the outline of the paper because the $h-$vector of $A(G)$ is equal to the $h-$vector of Δ_{P_n}. As it was proved in [3, Proposition 1.2], the $K-$graded algebra $\overline{A}(G)$ and the order complex $\Delta(I(P_n))$ have the same $h-$vector. (See [1], [5, Section 3] for the definition and the properties of the basic cover algebra associated to a graph and [2, §5.1] for the definition and the properties of the order complex of a poset).

For each subset $F \subset [n]$, $F \neq [n]$ we denote by $P_n(\overline{F})$ the subposet of P_n induced by the subset $\{p_i \mid i \notin F\}$ and by $G_{\overline{F}}$ the bipartite graph that comes from $P_n(\overline{F})$. Let $\Delta(I(P_n(\overline{F})))$ be the order complex of the distributive lattice $I(P_n(\overline{F}))$. If $F = [n]$ then, by convention, $\Delta(I(P_n(\overline{F})))=\emptyset$.

The main result of this paper is given in Theorem 2.2, which proves that one may reduce the computation of the $f-$ and $h-$vectors of the simplicial complex Δ_{P_n} and, consequently, the $h-$vector of $A(G)$ to the computation of $f^{\overline{F}}-$ and $h^{\overline{F}}-$vectors of the simplicial complex $\Delta(I(P_n(\overline{F})))$ and, consequently, the $h-$vector of the basic cover algebra $\overline{A}(G_{\overline{F}})$, for all $F \subset [n]$. Namely, we get the following formulas:

$$f_{j-1} = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} f_{j-l-1}^{\overline{F}}, \text{ for all } 1 \leq j \leq n+1,$$

$$h_j = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} h_{j-l}^{\overline{F}}, \text{ for all } 1 \leq j \leq n+1.$$

2. THE $h-$VECTOR AND THE HILBERT FUNCTION OF VERTEX COVER ALGEBRAS OF COHEN-MACAULAY BIPARTITE GRAPHS

In the first part of this section, we recall some definitions and results concerning the toric ideal Q_G of the vertex cover algebra $A(G)$ as they were given in [4]. That will allow us to introduce the simplicial complex Δ_{P_n}, which was already mentioned in the previous section.

Let $S = K[x_1, x_2, ..., x_n, y_1, y_2, ..., y_n]$ be the polynomial ring in $2n$ variables over a field K and let $G = G(P_n)$, where $P_n = \{p_1, p_2, ..., p_n\}$ is a poset such that $p_i \leq p_j$ implies $i \leq j$. For each $C \in M(G)$, we denote $m_C = \left(\prod_{x_i \in C} x_i\right) \cdot \left(\prod_{y_j \in C} y_j\right)$. Since G is Cohen-Macaulay, it is also unmixed, hence $|C| = n$ and $\deg m_C = n$, for all $C \in M(G)$.

We denote $B_G = K[\{x_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq n}, \{u_\alpha\}_{\alpha \in I(P_n)}]$. The *toric ideal* Q_G of $A(G)$ is the kernel of the surjective homomorphism $\Phi : B_G \to A(G)$ defined by $\Phi(x_i) = x_i$, $\Phi(y_j) = y_j$, $\Phi(u_\alpha) = m_\alpha t$, where $m_\alpha = \left(\prod_{p_i \in \alpha} x_i\right) \cdot \left(\prod_{p_j \notin \alpha} y_j\right)$.

Let $<_{lex}$ denote the lexicographic order on $K[\{x_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq n}]$ induced by the ordering $x_1 > x_2 > ... > x_n > y_1 > y_2 > ... > y_n$ and $<^{\#}$ the reverse lexicographic order on $K[\{u_\alpha\}_{\alpha \in I(P_n)}]$ induced by an ordering of the variables u_α's such that $u_\alpha > u_\beta$ if $\beta \subset \alpha$ in $I(P_n)$. Herzog and Hibi introduced in [4] the new monomial order $<_{lex}^{\#}$ on B_G defined as the product of the monomial orders

$<_{lex}$ and $<^{\#}$ from above. The reduced Gröbner basis Gr of the toric ideal Q_G of $A(G)$ with respect to the monomial order $<_{lex}^{\#}$ on B_G was computed in [4, Theorem 1.1]:

$$Gr = \{\underline{x_j u_\alpha} - y_j u_{\alpha\cup\{p_j\}}, j \in [n], \alpha \in I(P_n), p_j \notin \alpha, \alpha \cup \{p_j\} \in I(P_n),$$
$$\underline{u_\alpha u_\beta} - u_{\alpha\cup\beta} u_{\alpha\cap\beta}, \alpha, \beta \in P_n, \alpha \not\subset \beta, \beta \not\subset \alpha\},$$

where the initial monomial of each binomial of Gr is the first monomial.

Let Δ_{P_n} be the simplicial complex whose Stanley-Reisner ideal $I_{\Delta_{P_n}}$ coincides with $in_{<_{lex}^{\#}}(Q_G)$. Thus Δ_{P_n} is the simplicial complex on the set $[n] \bigcup I(P_n)$ whose faces are:

$$F \bigcup (L \setminus \{\alpha \in L \mid (\exists) j \in F \text{ such that } p_j \notin \alpha \text{ and } \alpha \cup \{p_j\} \in I(P_n)\}),$$

where $F \subset [n]$ and L is a chain of $I(P_n)$.

In order to identify the facets of Δ_{P_n} we need to make the following remark.

Because $I(P_n)$ is a full sublattice of the Boolean lattice BL_n on the set $\{p_1, p_2, ..., p_n\}$, ([5, Theorem 2.2]), then for each maximal chain L_m of $I(P_n)$ and for each $p_i \in P_n$, $1 \le i \le n$, there is a unique poset ideal $\alpha_{i,L_m} \in L_m$ such that $p_i \notin \alpha_{i,L_m}$ and $\alpha_{i,L_m} \cup \{p_i\} \in L_m$. Moreover, if $p_i \ne p_j$, then $\alpha_{i,L_m} \ne \alpha_{j.L_m}$ and $\{\alpha_{i_1,L_m}, ..., \alpha_{i_k,L_m}\} \subset \{\alpha \in L_m \mid (\exists) j \in F \text{ such that } p_j \notin \alpha \text{ and } \alpha \cup \{p_j\} \in I(P_n)\}$.

Therefore, the facets of Δ_{P_n} are either the maximal chains L_m of $I(P_n)$ or the faces of the form

$$F \bigcup (L_m \setminus \{\alpha_{i.L_m} \mid i \in F\}), \emptyset \ne F \subset [n],$$

with the property that

$$\{\alpha \in L_m \mid (\exists) j \in F \text{ such that } p_j \notin \alpha \text{ and } \alpha \cup \{p_j\} \in I(P_n)\} = \{\alpha_{i,L_m} \mid i \in F\}.$$

Since all maximal chain of $I(P_n)$ have the same length n, it follows that Δ_{P_n} is a pure simplicial complex of $\dim \Delta_{P_n} = n$.

Let us recall the poset $P_3 = \{p_1, p_2, p_3\}$ from the Example 1.1. Then Δ_{P_3} is a 3-dimensional simplicial complex on $[3] \bigcup I(P_3)$. Let L_1 be the maximal chain $\{\emptyset, \{p_1\}, \{p_1, p_2\}, \{p_1, p_2, p_3\}\}$ of $I(P_3)$. Therefore $\alpha_{1,L_1} = \emptyset$, $\alpha_{2,L_1} = \{p_1\}$ and $\alpha_{3,L_1} = \{p_1, p_2\}$.

If we put $F = \{1,2\}$, then

$$\{\alpha \in L_1 \mid (\exists) j \in \{1,2\} \text{ such that } p_j \notin \alpha \text{ and } \alpha \cup \{p_j\} \in I(P_3)\} = \{\emptyset, \{p_1\}\}$$

and thus $E_1 = \{1,2\} \bigcup \{\{p_1, p_2\}, \{p_1, p_2, p_3\}\}$ is a facet of the simplicial complex Δ_{P_3}, since $\{\alpha_{1,L_1}, \alpha_{2,L_2}\} = \{\emptyset, \{p_1\}\}$.

If we put $F = \{1,3\}$, then

$$\{\alpha \in L_1 \mid (\exists) j \in \{1,3\} \text{ such that } p_j \notin \alpha \text{ and } \alpha \cup \{p_j\} \in I(P_3)\} = \{\emptyset, \{p_1\}, \{p_1, p_2\}\}$$

and thus $E_2 = \{1,3\} \bigcup \{\{p_1, p_2, p_3\}\}$ is a face of the simplicial complex Δ_{P_3}, but it is not a facet of Δ_{P_3}, since $\{\alpha_{1,L_1}, \alpha_{3,L_1}\} \subset \{\emptyset, \{p_1\}, \{p_1, p_2\}\}$. We notice that E_2 is contained in the maximal face (facet) $E_3 = \{1,2,3\} \bigcup \{\{p_1, p_2, p_3\}\}$.

Lemma 2.1. Let $P_n = \{p_1, p_2, ..., p_n\}$, $n \ge 1$, be a poset such that $p_i \le p_j$ implies $i \le j$. Let $E = F \bigcup L$ be a face of the simplicial complex Δ_{P_n}, where $F \subset [n]$ and $L \ne \emptyset$. If $\alpha, \beta \in L$ such that $\alpha \cap P_n(\overline{F}) = \beta \cap P_n(\overline{F})$, then $\alpha = \beta$.

Proof. If $F = \emptyset$, then $P_n(\overline{F}) = P_n$, whence $\alpha = \beta$.

If $F = [n]$, then, by the definition of Δ_{P_n}, it follows that $L = \{p_1, p_2, ..., p_n\}$. Hence $\alpha = \beta$.

We may assume $\emptyset \subset F \subset [n]$. We show that $\beta \subseteq \alpha$ and $\alpha \subseteq \beta$.

Let us suppose, on the contrary, that $\beta \not\subset \alpha$. Then there is some $p_{r_1} \in \beta \setminus \alpha$. If $r_1 \notin F$, then $p_{r_1} \in \beta \cap P_n(\overline{F})$. Since $\alpha \cap P_n(\overline{F}) = \beta \cap P_n(\overline{F})$, it follows that $p_{r_1} \in \alpha$, which is a contradiction to the choice of p_{r_1}. Hence $r_1 \in F$ and $p_{r_1} \notin \alpha$. By the definition of Δ_{P_n}, if $\alpha \in L$ and $p_{r_1} \notin \alpha$, then $\alpha \cup \{p_{r_1}\} \notin I(P_n)$, which implies that there is $p_{r_2} \in P_n$ such that $p_{r_2} \le p_{r_1}$ and $p_{r_2} \notin \alpha \cup \{p_{r_1}\}$, that means $p_{r_2} \notin \alpha$ and $p_{r_2} \ne p_{r_1}$. Since k, $p_{r_1} \in \beta$ and $p_{r_2} \le p_{r_1}$, it follows that $p_{r_2} \in \beta$. Thus $p_{r_2} \in \beta \setminus \alpha$.

By repeated application of this argument, we get the following strictly decreasing sequence $... < p_{r_{k+1}} < p_{r_k} < ... < p_{r_2} < p_{r_1}$, where $p_{r_k} \in \beta \setminus \alpha$, for all $k \ge 1$. The sequence is not stationary, hence the set $\{..., p_{r_{k+1}}, p_{r_k}, ..., p_{r_2}, p_{r_1}\}$ is infinite, which is a contradiction, since $\{..., p_{r_{k+1}}, p_{r_k}, ..., p_{r_2}, p_{r_1}\} \subset \beta \setminus \alpha \subset \{p_1, p_2, ..., p_n\}$. Hence $\beta \subseteq \alpha$.

Similarly, we can show that $\alpha \subseteq \beta$. Hence $\alpha = \beta$.

The main result of the paper relates the $h-$vector of $A(G)$ to the $h-$vector of $\overline{A}(G_{\overline{F}})$, for all $F \subset [n]$. If $F = [n]$, then, by convention, $f^{\overline{[n]}}$ and $h^{\overline{[n]}}$ are the $f-$ and $h-$vectors of the order complex $\Delta(I(P_n(\overline{[n]}))) = \{\emptyset\}$.

Theorem 2.2. Let $f = (f_0, f_1, ..., f_n)$ and $h = (h_0, h_1, ..., h_n, h_{n+1})$, respectively, $f^{\overline{[F]}}$ and $h^{\overline{[F]}}$ be the $f-$ and $h-$vectors of the simplicial complex Δ_{P_n} and, consequently, the $h-$vector of $A(G)$, respectively, of the simplicial complex $\Delta(I(P_n(\overline{F})))$ and, consequently, the $h-$vector of $\overline{A}(G_{\overline{F}})$, for all $F \subset [n]$. Then the following relations hold:

$$f_{j-1} = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} f_{j-l-1}^{\overline{F}}, \text{ for all } 1 \le j \le n+1,$$

$$h_j = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} h_{j-l}^{\overline{F}}, \text{ for all } 1 \le j \le n+1.$$

Proof. The proof of the theorem is quite technical. In order to establish which are the main steps, we begin with a sketch of the proof. The main idea is to define a bijective map from the set of all $(j-1)-$dimensional faces $E \in \Delta_{P_n}$ of the form $E = F \bigcup L$, with $F \subset [n]$, $|F| = l$, L a chain of $\Delta(I(P_n))$ of length $j-l-1$, on the set of all couples $(F, L(\overline{F}))$, with $F \subset [n]$, $|F| = l$, $L(\overline{F})$ a chain of $\Delta(I(P_n(\overline{F})))$ of length $j-l-1$, for each $1 \le j \le n-1$ and for each $0 \le l \le \min\{j, n\}$. We denote this bijection by $\lambda = \lambda(j,l)$.

In the first step, we show that λ is well-defined. Secondly, we prove that λ is injective, which is essentially based on Lemma 2.1. Finally, in the most technical part of our proof, we show that λ is surjective.

The map λ is defined as follows:

if $E = F \cup L$ is a face of Δ_{P_n} that satisfies all required conditions, then $\lambda(E) = (F, L(\overline{F}))$, where

(i) $L(\overline{F}) = \emptyset$, if $L = \emptyset$;

(ii) $L(\overline{F}) = \{\emptyset\}$, if $L \neq \emptyset$ and $F = [n]$;

(iii) $L(\overline{F}) = \{\alpha \cap P_n(\overline{F}) \mid \alpha \in L\}$, if $L \neq \emptyset$ and $F \neq [n]$.

Step 1. We show that λ is well-defined.

If $E = F$, then $L = \emptyset$. In this case $L(\overline{F}) = \emptyset \in \Delta(I(P_n(\overline{F})))$ and both L and $L(\overline{F})$ are chains of length -1.

If $E \neq F$, then $L \neq \emptyset$. Put $L = \{\alpha_{t_1}, \alpha_{t_2}, ..., \alpha_{t_{j-l}}\}$ with $\alpha_{t_1} \subset \alpha_{t_2} \subset ... \subset \alpha_{t_{j-l}}$. Let $\beta_{t_i} = \alpha_{t_i} \cap P_n(\overline{F})$, for all $1 \leq i \leq j-l$. Then $L(\overline{F}) = \{\beta_{t_i} \mid 1 \leq i \leq j-l\}$.

We claim that $\beta_{t_i} \in I(P_n(\overline{F}))$, for all $1 \leq i \leq j-l$.

If $F = [n]$, then $l = n$, $j = l+1$ and $L = \{\{p_1, p_2, ..., p_n\}\}$. In this case $L(\overline{F}) = \{\emptyset\}$, hence $L(\overline{F})$ is a chain of $I(P_n(\overline{F}))$ of length 0.

Now, we may assume that $F \neq [n]$. Let $p_a \in \beta_{t_i}$ and $p_b \in P_n(\overline{F})$ with $p_a \leq p_b$ in $P_n(\overline{F})$. Obviously, $p_a \leq p_b$ in P_n. Since $\alpha_{t_i} \in I(P_n)$ and $p_a \in \beta_{t_i}$, it follows that $p_b \in \alpha_{t_i}$. Hence $p_b \in \beta_{t_i}$, which shows that $\beta_{t_i} \in I(P_n(\overline{F}))$, for all $1 \leq i \leq j-l$. Obviously, $\beta_{t_i} \subset \beta_{t_{i+1}}$, for all $1 \leq i \leq j-l-1$.

Now, we prove that $L(\overline{F}) = \{\beta_{t_1}, \beta_{t_2}, ..., \beta_{t_{j-l}}\}$, $\beta_{t_1} \subset \beta_{t_2} \subset ... \subset \beta_{t_{j-l}}$, is a chain of $I(P_n(\overline{F}))$ of length $j-l-1$. Let us suppose that there is some $1 \leq i \leq j-l-1$ such that $\beta_{t_i} = \beta_{t_{i+1}}$. Since $\beta_{t_i} = \alpha_{t_i} \cap P_n(\overline{F})$, $\beta_{t_{i+1}} = \alpha_{t_{i+1}} \cap P_n(\overline{F})$, it follows, by Lemma 2.1., that $\alpha_{t_i} = \alpha_{t_{i+1}}$, which is a contradiction. Hence $\beta_{t_i} \subset \beta_{t_{i+1}}$, for all $1 \leq i \leq j-l-1$.

Step 2. We show that λ is injective.

Let E and E' two faces of Δ_{P_n} with $E = F \cup L$, $E' = F' \cup L'$, $|E| = j$, $|E'| = j'$, $1 \leq j, j' \leq n+1$, $F \subset [n]$, $F' \subset [n]$, $|F| = l$, $0 \leq l \leq \min\{j, n\}$, $|F'| = l'$, $0 \leq l' \leq \min\{j', n\}$, L and L' chains of $\Delta(I(P_n))$ of length $j-l-1$, respectively, $j'-l'-1$, such that $\lambda(E) = \lambda(E')$. We prove that $E = E'$.

Since $(F, L(\overline{F})) = (F', L'(\overline{F'}))$, it follows that $F = F'$, $L(\overline{F}) = L'(\overline{F'}) = L'(\overline{F})$, $l = l'$ and $j = j'$.

If $L = \emptyset$, then $E = F$ and $j = l$. Hence $L' = \emptyset$ and $E = E'$.

Now, we may assume that $L \neq \emptyset$, which implies that $j > l$ and $L' \neq \emptyset$. Put $L' = \{\alpha'_{t_1}, \alpha'_{t_2}, ..., \alpha'_{t_{j-l}}\}$ with $\alpha'_{t_1} \subset \alpha'_{t_2} \subset ... \subset \alpha'_{t_{j-l}}$. Then $L'(\overline{F}) = \{\alpha'_{t_i} \cap P_n(F) \mid 1 \leq i \leq j-l\}$. Since $L(\overline{F})$ and $L'(\overline{F})$ are chains and $L(\overline{F}) = L'(\overline{F})$, it follows that $\alpha_{t_i} \cap P_n(\overline{F}) = \alpha'_{t_i} \cap P_n(F)$, for all $1 \leq i \leq j-l$. By Lemma 2.1. $\alpha_{t_i} = \alpha'_{t_i}$, for all $1 \leq i \leq j-l$, which implies that $L = L'$.

Hence $E = F \cup L = F' \cup L' = E'$.

Step 3. We show that λ is surjective.

Let $\left(L, L\left(\overline{F}\right)\right)$ be a couple such that $F \subset [n]$, $|F| = l$, $0 \le l \le \min\{j, n\}$ and $L\left(\overline{F}\right)$ is a chain of $I\left(P_n\left(\overline{F}\right)\right)$ of length $j - l - 1$.

If $L\left(\overline{F}\right) = \emptyset$, then $j = l$. Put $L = \emptyset$ and $E = F \bigcup L$. Then E is a face of Δ_{P_n} with $|E| = j$, L is a chain $\Delta(I(P_n))$ of length -1 and $\lambda(E) = \left(F, L\left(\overline{F}\right)\right)$.

We may assume that $L\left(\overline{F}\right) \ne \emptyset$.

If $F = \emptyset$, then $l = 0$ and $P_n\left(\overline{F}\right) = P_n$. Put $L = L\left(\overline{F}\right) \in \Delta(I(P_n))$ and $E = F \bigcup L$. Then E is a face of Δ_{P_n} with $|E| = j$ and $\lambda(E) = \left(F, L\left(\overline{F}\right)\right)$.

If $F = [\mathrm{n}]$, then $\Delta\left(I\left(P_n\left(\overline{F}\right)\right)\right) = \{\emptyset\}$, $l = n$ and $j = n + 1$. Hence $L\left(\overline{F}\right) = \{\emptyset\}$. Put $L = \{\{p_1, p_2, ..., p_n\}\}$ and $E = F \bigcup L$. Then E is a face of Δ_{P_n} with $|E| = j$ and $\lambda(E) = \left(F, L\left(\overline{F}\right)\right)$.

Therefore, we may assume that $\emptyset \subset F \subset [n]$ and $L\left(\overline{F}\right) = \left\{\beta_{t_1}, \beta_{t_2}, ..., \beta_{t_{j-l}}\right\}$ with $\beta_{t_1} \subset \beta_{t_2} \subset ... \subset \beta_{t_{j-l}}$. We show that for each $1 \le i \le j - l$, there is some subset $\gamma_{t_i} \subset \{p_u \mid u \in F\}$ such that $\beta_{t_i} \cup \gamma_{t_i} \in I(P_n)$. Namely, if we choose $\gamma_{t_i} = \left\{p_u \mid u \in F, (\exists) p_v \in \beta_{t_i}\ with\, p_u \le p_v\right\}$, $\beta_{t_i} \cup \gamma_{t_i} \in I(P_n)$, for all $1 \le i \le j - l$.

Indeed, let $p_a \in \beta_{t_i} \cup \gamma_{t_i}$ and $p_b \in P_n$ with $p_b \le p_a$. We must analyze the following cases:

Case 1. $p_a \in \beta_{t_i}$. If $b \in F$, then $p_b \in \gamma_{t_i}$. If $b \in \overline{F}$, then $p_b \in P_n\left(\overline{F}\right)$ and $p_b \le p_a$ in $P_n\left(\overline{F}\right)$. Since $\beta_{t_i} \in I\left(P_n\left|\overline{F}\right|\right)$, it follows that $p_b \in \beta_{t_i}$.

Case 2. $p_a \in \gamma_{t_i}$. By the definition of γ_{t_i}, there is some $p_c \in \beta_{t_i}$ with $p_a \le p_c$. Therefore $p_b \le p_c$. If $b \in F$, then $p_b \in \gamma_{t_i}$. If $b \notin F$, then $p_b \in P_n\left(\overline{F}\right)$ and $p_b \le p_c$ in $P_n\left(\overline{F}\right)$. Since $\beta_{t_i} \in I\left(P_n\left(\overline{F}\right)\right)$, it follows that $p_b \in \beta_{t_i}$.

For each $1 \le i \le j - l$, we choose $\delta_{t_i} \subset \{p_u \mid u \in F\}$ to be a maximal subset such that $\gamma_{t_i} \subset \delta_{t_i}$ and $\beta_{t_i} \cup \delta_{t_i} \in I(P_n)$. We put $\alpha_{t_i} = \beta_{t_i} \cup \delta_{t_i}$, $1 \le i \le j - l$, and $L = \left\{\alpha_{t_1}, \alpha_{t_2}, ..., \alpha_{t_{j-l}}\right\}$.

We claim that $\alpha_{t_1} \subset \alpha_{t_2} \subset ... \subset \alpha_{t_{j-l}}$, which implies that $L \in \Delta(I(P_n))$ and the length of L is exactly $j - l - 1$.

We show that $\alpha_{t_i} \subset \alpha_{t_{i+1}}$, for all $1 \le i \le j - l - 1$.

Let us suppose, on the contrary, that there is some $1 \le i \le j - l - 1$ with $\alpha_{t_i} \not\subset \alpha_{t_{i+1}}$. Then there is some $p_{r_1} \in \alpha_{t_i} \setminus \alpha_{t_{i+1}}$. Obviously, $p_{r_1} \in \{p_u \mid u \in F\}$, otherwise, if $p_{r_1} \in P_n\left(\overline{F}\right)$, then $p_{r_1} \in \beta_{t_i} \subset \beta_{t_{i+1}}$. Since $\alpha_{t_{i+1}} = \beta_{t_{i+1}} \cup \delta_{t_{i+1}}$, it follows that $p_{r_1} \in \alpha_{t_{i+1}}$, which is a contradiction.

By the choice of $\delta_{t_{i+1}}$, $\alpha_{t_{i+1}} \in I(P_n)$ and $\alpha_{t_{i+1}} \cup \{p_{r_1}\} \notin I(P_n)$. Then there is some $p_{r_2} \in P_n$ such that $p_{r_2} \le p_{r_1}$ and $p_{r_2} \notin \alpha_{t_{i+1}} \cup \{p_{r_1}\}$, which implies that $p_{r_2} \notin \alpha_{t_{i+1}}$ and $p_{r_1} \ne p_{r_2}$. Hence $\alpha_{t_i} \in I(P_n)$, $p_{r_1} \in \alpha_{t_i}$ and $p_{r_2} < p_{r_1}$, which implies that $p_{r_2} \in \alpha_{t_i}$.

By repeated application of this argument, we get an infinite sequence strictly decreasing $... < p_{r_{k+1}} < p_{r_k} < ... < p_{r_2} < p_{r_1}$, where $p_{r_k} \in \alpha_{t_i} \setminus \alpha_{t_{i+1}}$, for all $k \ge 1$. The sequence is not stationary, hence the $\left\{..., p_{r_{k+1}}, p_{r_k}, ..., p_{r_2}, p_{r_1}\right\}$ is infinite, which is a contradiction.

We obviously get $\alpha_{t_i} \subset \alpha_{t_{i+1}}$, for all $1 \le i \le j-l-1$. If there is some $1 \le i \le j-l-1$ with $\alpha_{t_i} = \alpha_{t_{i+1}}$, then $\beta_{t_i} = \beta_{t_{i+1}}$, which is a contradiction.

Put $L = \{\alpha_{t_1}, \alpha_{t_2}, ..., \alpha_{t_{j-l}}\}$ and $E = F \bigcup L$. We notice that, for each $p_k \in F$ and for each $\alpha \in L$, either $p_k \in \alpha$, or, if $p_k \notin \alpha$, then $\alpha \cup \{p_k\} \notin I(P_n)$. Hence E is a face of Δ_{P_n} with $|E| = j$ and $\lambda(E) = (F, L(\overline{F}))$.

By the definition of Δ_{P_n}, the vertex cover algebra $A(G)$ and the simplicial complex Δ_{P_n} have the same $f-$ and $h-$vectors. As we already noticed in Remark 3.2, the basic vertex cover algebra $\overline{A}(G_{\overline{F}})$ and the simplicial complex $\Delta(I(P_n(\overline{F})))$ have the same $f-$ and $h-$vectors, for all $F \subset [n]$.

Hence, by using the bijective map λ, we get:

$$f_{j-1} = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} f_{j-l-1}^{\overline{F}}, \text{ for all } 1 \le j \le n+1.$$

By the formulas that relate the $h-$vector to the $f-$vector of a simplicial complex (see [2, Lemma 5.1.8], we get:

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{n+1-i}{j-i} f_{i-1} = \sum_{i=0}^{j} (-1)^{j-i} \binom{n+1-i}{j-i} \cdot \left(\sum_{l=0}^{i} \sum_{\substack{F \subset [n] \\ |F| = l}} f_{i-l-1}^{\overline{F}} \right) =$$

$$= \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} \sum_{i=0}^{j-l} (-1)^{j-l-i} \binom{n+1-l-i}{j-l-i} f_{i-1}^{\overline{F}} = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} h_{j-l}^{\overline{F}}, \text{ for all } 1 \le j \le n+1.$$

Let $P_3 = \{p_1, p_2, p_3\}$ be the poset from Example 1.1. Then the simplicial complex Δ_{P_3} on the set $[3] \bigcup I(P_3)$ has the $f-$vector $(9,26,30,12)$ (i.e. $f_0 = 9$, $f_1 = 26$, $f_2 = 30$, $f_3 = 12$).

Thus, the Hilbert series of $A(G)$ is:

$$H_{A(G)}(z) = \frac{f_{-1}(1-z)^4 + f_0 z(1-z)^3 + f_1 z^2 (1-z)^2 + f_2 z^3 (1-z) + f_3 z^4}{(1-z)^7} =$$

$$= \frac{(1-z)^4 + 9z(1-z)^3 + 26z^2(1-z)^2 + 30z^3(1-z) + 12z^4}{(1-z)^7} = \frac{z^3 + 5z^2 + 5z + 1}{(1-z)^7}.$$

Hence the $h-$vector of $A(G)$ is $(1,5,5,1,0)$ (i.e. $h_0 = h_3 = 1$, $h_1 = h_2 = 5$, $h_4 = 0$).

Finally, the Hilbert function of $A(G)$ is:

$$H(A(G), k) = h_0 \cdot \binom{6+k}{k} + h_1 \cdot \binom{5+k}{k-1} + h_2 \cdot \binom{4+k}{k-2} + h_3 \cdot \binom{3+k}{k-3} =$$

$$= \binom{6+k}{k} + 5 \cdot \binom{5+k}{k-1} + 5 \cdot \binom{4+k}{k-2} + \binom{3+k}{k-3} = \frac{1}{60}k^6 + \frac{1}{5}k^5 + k^4 + \frac{8}{3}k^3 + \frac{239}{60}k^2 + \frac{47}{15}k + 1$$

for all $k \ge 0$.

References

1. Benedetti B., Constantinescu A., Varbaro M., Dimension, depth and zero-divisors of the basic $k-$covers of a graph, Le Matematiche, Volume LXIII, Issue II (2009), 117–156.
2. Bruns W., Herzog J., Cohen-Macaulay Rings, rev. ed., Cambridge Stud. Adv. Math., 39, Cambridge Univ. Press, Cambridge, 1998.
3. Ion C., On the Hilbert series of vertex cover algebras of Cohen-Macauley bipartite graphs, Le Matematiche, Volume LXV, Issue I (2010), 89-99.
4. Herzog J., Hibi T., Distributive lattices, Bipartite Graphs and Alexander Duality, J. Algebraic Combin. 22 (2005), 289–302.
5. Herzog J., Hibi T., Ohsugi H., Unmixed bipartite graphs and sublattices of the Boolean lattices, J. Algebraic Combin. 30 (2009), 415–420.
6. Herzog J., Hibi T., Trung N. V., Symbolic powers of monomial ideals and vertex cover algebras, Adv. Math. 210 (2007), 304–322.