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The use of interpolation polynomials in the approximation of functions: comparative study

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Abstract
In this study, we present three methods for approximating functions using interpolation polynomials. In engineering, there are many situations where it is necessary to approximate the value of a function at a certain point, knowing only a finite set of experimental data or when the function itself is presented in a form that is not easily usable. We use specific methods in numerical analysis such as Lagrange interpolation, the method of dividing differences and Neville’s method. Given a function f defined on an interval [a,b], the purpose of this work is to construct a function that approximates the function f at a predetermined value. The expression of the approximation functions will be calculated and we will evaluate the obtained error for each approximation.

Keywords: Polynomial approximation, Lagrange interpolating polynomial, Newton’s method, Neville’s method.

1. INTRODUCTION

Polynomial interpolation is a method often used in engineering for modeling different processes [10], or when processing an experimental data set. There is a concern to evaluate the efficiency of interpolation polynomials using different interpolation methods [4,8].
The aim of this work is to calculate an interpolation polynomial that will give us the possibility of approximating the function at a certain value. The work focuses on evaluating function approximations using the Lagrange interpolation method [9], the Newton interpolation method [9] and Neville’s interpolation method [6].

Let f be the function defined on $\mathbb{R}$ with values on $\mathbb{R}$

$$f(x) = xe^{-x} \tag{1}$$

We aim to approximate the value of $f(2.2)$ using the five distinct values given in Table 1. To streamline the calculations in this note, we will use the first five decimal places of each result without rounding.

Table 1. Data to be used in evaluation of $f(2.2)$

x	$x_0 = 1$	$x_1 = 1.5$	$x_2 = 2$	$x_3 = 2.5$	$x_4 = 3$
$f(x)$	0.36787	0.33469	0.27067	0.20521	0.14936

2. RESULTS AND DISCUSSION

We aim to find the interpolation polynomial to use in the evaluation of $f(2.2)$. We present below the results obtained using/ through the three chosen interpolation methods.

Lagrange interpolation method

The first method analyzed for constructing the interpolation polynomial to calculate the value for $x = 2.2$ is Lagrange's method. Thus, we use theorem 3.2 from [1 page 110], and obtain

$$P(x) = \sum_{k=0}^{4} f(x_k) L_k(x) \tag{2}$$

where $L_k(x)$ is given by the formula:

$$L_k(x) = \prod_{i=0, i \neq k}^{4} \frac{(x - x_i)}{(x_k - x_i)}, 0 \leq k \leq 4. \tag{3}$$

Using formula (3) we can calculate the basic Lagrange polynomials $L_0(x), L_1(x), L_2(x), L_3(x)$ and $L_{4(x)}$. Thus, the following are obtained:

$$L_0(x) = \frac{(x-1.5)(x-2)(x-2.5)(x-3)}{(1-1.5)(1-2)(1-2.5)(1-3)} = \frac{2}{3}(x-1.5)(x-2)(x-2.5)(x-3) \tag{4}$$

$$L_1(x) = \frac{(x-1)(x-2)(x-2.5)(x-3)}{(1.5-1)(1.5-2)(1.5-2.5)(1.5-3)} = -\frac{8}{3}(x-1)(x-2)(x-2.5)(x-3) \tag{5}$$

$$L_2(x) = \frac{(x-1)(x-1.5)(x-2.5)(x-3)}{(2-1)(2-1.5)(2-2.5)(2-3)} = 4(x-1)(x-1.5)(x-2.5)(x-3) \tag{6}$$

$$L_3(x) = \frac{(x-1)(x-1.5)(x-2)(x-3)}{(2.5-1)(2.5-1.5)(2.5-2)(2.5-3)} = -\frac{8}{3}(x-1)(x-1.5)(x-2)(x-3) \tag{7}$$

$$L_4(x) = \frac{(x-1)(x-1.5)(x-2)(x-2.5)}{(3-1)(3-1.5)(3-2)(3-2.5)} = \frac{2}{3}(x-1)(x-1.5)(x-2)(x-2.5) \tag{8}$$

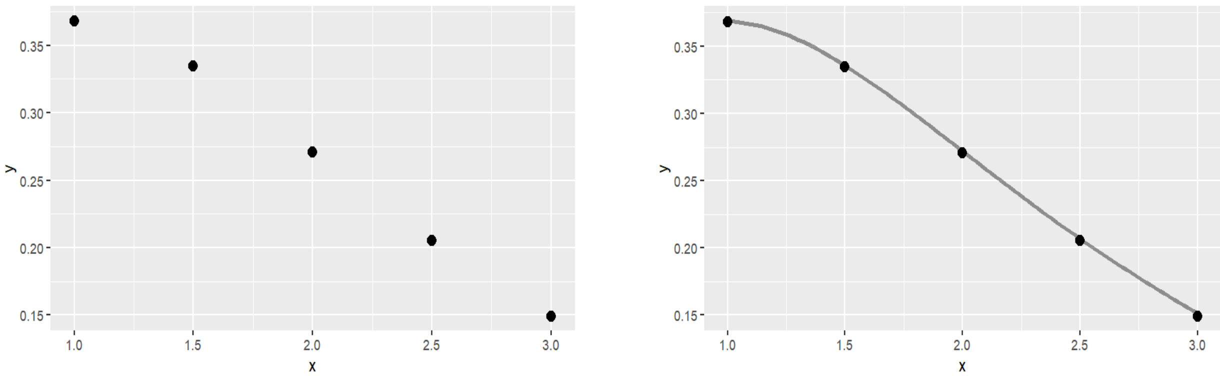
Substituting the results obtained in formula (2), we obtain the interpolation polynomial in Lagrange form:

$$P(x) = -0.01223x^4 + 0.12480x^3 - 0.45517x^2 + 0.57810x + 0.13235 \tag{9}$$

Thus, an approximation of the $f(2.2)$ value, see Fig. 2, is:

$$f(2.2) \approx P(2.2) = 0.24352$$

In order to perform a visual inspection of the initial values and the calculated interpolation polynomial, we graphically represented the initial values, Fig. 1a), and the interpolation polynomial, Fig. 1b). It can be observed that the interpolation polynomial in Lagrange form passes through the points representing the initial values.



a) *b)*

Fig.1 a) Graphic representation of the initial values. b) The graphic representation of the interpolation polynomial in relation to the initial values [11]

The relative error is:

$$\frac{|f(2.2) - P(2.2)|}{|f(2.2)|} = \frac{|0.24376 - 0.24352|}{|0.24376|} = 0.00098 \tag{10}$$

Absolute error is:

$$|f(2.2) - P(2.2)| = 0.00024 \tag{11}$$

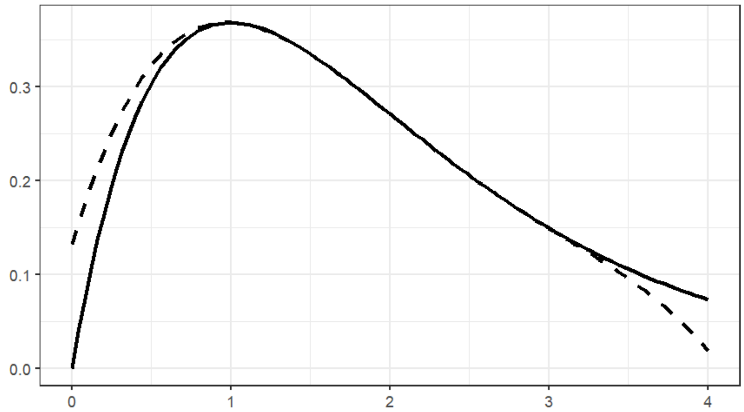


Fig. 2. The function (solid curve) vs. the interpolation polynomial (dashed curve) [11]

To calculate the interpolation error $f(x) - P(x)$, we apply theorem 3.4 from [7 page 152], thus we obtain:

$$f(x) - P(x) = \frac{(x-1)(x-1.5)(x-2)(x-2.5)(x-3)}{5!} f^{(5)}(c) \tag{12}$$

where $c \in [1,3]$.

Therefore, we obtain the vertical distance between $y = xe^{-x}$ and $y = P(x)$, the interpolation polynomial, from Fig. 2. To calculate the derivative of the function $f(x) = xe^{-x}$ of the fifth order is obtained:

$$f(x) = xe^{-x}, f'(x) = e^{-x}(1-x), f''(x) = e^{-x}(x-2), f'''(x) = e^{-x}(3-x),$$
$$f''''(x) = e^{-x}(x-4),$$
$$f^{(5)}(x) = e^{-x}(5-x)$$

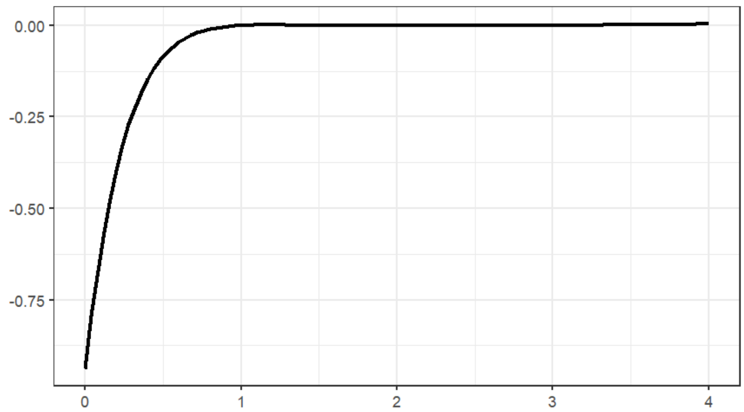


Fig. 3. Plot of the error bound [11]

But for $x \in [1,3]$ we have:

$$\left|f^{(5)}(x)\right| \leq \frac{4}{e}, x \in [1,3]$$

At $x = 2.2$, the interpolation error is obtained using formula (12).

$$|f(2.2) - P(2.2)| \leq \frac{(2.2-1)(2.2-1.5)(2.2-2)(2,2-2.5)(2.2-3)}{120} \cdot \frac{4}{e} \approx 0.00048$$

Thus, the maximum error $|f(2.2) - P(2.2)|$, or error bound, on the interval $[1,3]$ is 0.00048 (see Fig. 3).

Newton interpolation method

The second method used in the construction of the interpolation polynomial involves obtaining the interpolation polynomial/it in Newton form. We use the initial values, Table 1, to calculate the coefficients of the interpolation polynomial. We use formula (18.14) from [2 page 495], for the calculation of divided differences, as follows:

$$f[x_n, x_{n-1}, \dots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \dots, x_1] - f[x_{n-1}, x_{n-2}, \dots, x_0]}{x_n - x_0} \quad (13)$$

Thus, in the first step we will consider:

$$f[x_0] = 0.36787, f[x_1] = 0.33469, f[x_2] = 0.27067, f[x_3] = 0.20521, f[x_4] = 0.14936$$

We calculate the first divided differences:

$$f[x_0, x_1] = \frac{0.33469 - 0.36787}{1.5 - 1} = -0.06636$$

$$f[x_1, x_2] = \frac{0.27067 - 0.33469}{2 - 1.5} = -0.12804$$

$$f[x_2, x_3] = \frac{0.20521 - 0.27067}{2.5 - 2} = -0.13092$$

$$f[x_3, x_4] = \frac{0.14936 - 0.20521}{3 - 2.5} = -0.11170$$

The second divided differenceare:

$$f[x_0, x_1, x_2] = \frac{-0.12804 + 0.06636}{2 - 1} = -0.06168$$

$$f[x_1, x_2, x_3] = \frac{-0.13092 + 0.12804}{2.5 - 1.5} = -0.00288$$
$$f[x_2, x_3, x_4] = \frac{-0.11170 + 0.13092}{3 - 2} = 0.01922$$

The third divided differences are:

$$f[x_0, x_1, x_2, x_3] = \frac{-0.00288 + 0.06168}{2.5 - 1} = 0.03920$$
$$f[x_1, x_2, x_3, x_4] = \frac{0.01922 + 0.00288}{3 - 1.5} = 0.01473$$

The fourth divided differences are:

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{0.01473 - 0.03920}{3 - 1} = -0.01223$$

With the help of these differences calculated above we can obtain the expression of the interpolation polynomial in Newton form, applying theorem 4.2, from [3 page 177]:

$$P(x) = -0.01223x^4 + 0.12480x^3 - 0.45517x^2 + 0.57806x + 0.13238 \tag{14}$$

We can use the obtained polynomial to evaluate the $f(2.2)$, so:

$$f(2.2) \approx P(2.2) = 0.24346$$

The relative error is:

$$\frac{|f(2.2) - P(2.2)|}{|f(2.2)|} = \frac{|0.24376 - 0.24346|}{|0.24376|} = 0.00123 \tag{15}$$

The absolute error is:

$$|f(2.2) - P(2.2)| = 0.00030 \tag{16}$$

Neville’s interpolation method

The last method used in this article to evaluate $f(2.2)$ is the method developed by Neville. We use theorem 3.5, from [1 page 119], to define the expressions:

$$Q_{i,j} = \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}} \tag{17}$$

We consider $Q_{i,0} = f(x_i), i = \overline{0,4}$ (theorem 1.1.3, from [5 page 11]), so:

$$Q_{0,0} = 0.36787, Q_{1,0} = 0.33469, Q_{2,0} = 0.27067, Q_{3,0} = 0.20521, Q_{4,0} = 0.14936$$

Thus, using the formula given by (17) we can calculate $Q_{i,j}$:

$$Q_{1,1} = -0.06635x + 0.43423$$
$$Q_{2,1} = -0.12804x + 0.52675$$
$$Q_{3,1} = -0.13091x + 0.53251$$
$$Q_{4,1} = -0.11170x + 0.48446$$

$$Q_{2,2} = -0.06168x^2 + 0.08786x + 0.34170$$
$$Q_{3,2} = -0.00287x^2 - 0.11797x + 0.51811$$
$$Q_{4,2} = 0.01920x^2 - 0.21738x + 0.62861$$

$$Q_{3,3} = 0.03920x^3 - 0.23810x^2 + 0.34268x + 0.22408$$
$$Q_{4,3} = 0.01471x^3 - 0.09120x^2 + 0.05510x + 0.40760$$

Finally, the interpolation polynomial is obtained in Neville form:

$$P(x) = Q_{4,4}(x) = -0.01224x^4 + 0.12489x^3 - 0.45534x^2 + 0.57826x + 0.13228 \tag{18}$$

We can use the obtained polynomial to evaluate the $f(2.2)$, so:

$$f(2.2) \approx P(2.2) = 0.24370$$

The relative error is:

$$\frac{\left|f(2.2) - Q_{4,4}(2.2)\right|}{|f(2.2)|} = \frac{|0.24376 - 0.24370|}{|0.24376|} = 0.00024 \tag{19}$$

The absolute error is:

$$\left|f(2.2) - Q_{4,4}\,(2.2)\right| = 0.00006 \tag{20}$$

3. CONCLUSIONS

In this note we evaluated the efficiency of three interpolation methods: the Lagrange interpolation method, the Newton interpolation method and the Neville interpolation method. For each of the three methods presented, we calculated the expression of the interpolation polynomial used in obtaining the approximation of the function value in 2.2. In each case, the relative error, as well as the absolute error, was calculated. We note that, in this case, for the chosen function and the values from Table 1, the best results were obtained using the Neville interpolation method.

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