ANNALS OF “DUNAREA DE JOS” UNIVERSITY OF GALATI
MATHEMATICS, PHYSICS, THEORETICAL MECHANICS
FASCICLE II, YEAR XVI (XLVII) 2024, No. 2
DOI: https://doi.org/10.35219/ann-ugal-math-phys-mec.2024.2.01

Geometric characterization of V-harmonic maps

Monica Alice Aprodu[1,*]

[1] *“Dunarea de Jos” University of Galati, Faculty of Sciences and Environment, 111 Domneasca Str.,800201 Galati, Romania*
[*] *Corresponding author: maprodu@ugal.ro*

Abstract
In this paper, we provide a geometric characterization of a V-harmonic map and establish conditions that relate the V-harmonicity of a map to the V-minimality of its fibers.

Keywords: Harmonic maps, V-harmonic maps, V-minimal submanifolds

1. INTRODUCTION

Let $\varphi:M\rightarrow N$ be a smooth map between Riemannian manifolds, and let V be a smooth vector field on the source manifold. Chen, Jost, and Wang [4] introduced the notion of a V-harmonic map using the condition:

$$\tau_V(\varphi) := \tau(\varphi) + d\varphi(V) = 0,$$

where $\tau(\varphi)$ is the tension field. Zhao, see [Z], established the relationship between the notions of V-harmonicity of a map and the minimality of its fibers. The existence of the vector field V on the source manifold plays an important role in the definition of V-minimal submanifolds, a notion introduced in [1], which is a natural extension of the classical concept. A V-minimal submanifold is defined by:

$trace\ (A) - V$ is a tangent vector field to the submanifold,

where A is the second fundamental form of the submanifold.

As in the case of V-harmonicity, where for $V = 0$, V-harmonicity and harmonicity coincide, V-minimality is equivalent to minimality when the vector field V is zero.

The outline of the paper is as follows. First, we recall some basic notions needed for our study, including the second fundamental form of a submanifold, horizontally conformal maps, mean curvature, and V-harmonic maps, along with some of their properties. Then, we give a geometric characterization of V-harmonic maps (Theorem 3.3) and we establish the relationship between V-harmonicity of a map and the V-minimality of its fibers(Theorem 3.2).

2. V- HARMONIC MAPS

In this section we recall basic facts about V-harmonic maps and their properties, which will be used in the sequel.
Let (M^m, g) and (N^n, h) be two Riemannian manifolds of dimensions m and n, respectively.

Considering a smooth function *f: M→* ***R***, *the gradient* of *f*, denoted by *grad (f)* or ∇f (see, for example, [3] or [9]), is the vector field characterized as follows: *g(grad f, X)=df (X)=X f*, for any vector field *X* in *M*.

Let *φ:M→N* be a smooth map between the Riemannian manifolds.
For the vector bundle $TN \to N$, the *pull-back bundle* $\varphi^{-1}TN \to M$, has fibers given by $(\varphi^{-1}TN)_x = T_xN, x \in M$. The connection on the pull-back bundle, denoted by ∇^φ, is the unique linear connection such that, for any section σ on the tangent bundle TN, $\nabla^\varphi_X(\sigma \circ \varphi) = \nabla^N_{d\varphi(X)}(\sigma)$, where $\sigma \circ \varphi$ is a section on the pull-back bundle $\varphi^{-1}TN$, ∇^N is the connection on the vector bundle $TN \to N$ (see example [3]). Considering the bundle $Hom\,(TM, \varphi^{-1}TN) \to M$ (where $d\varphi$ can be viewed as a section of it), it has a connection ∇ induced by the Levi-Civita connection of *M*, ∇^M, and the pull-back connection described above. Applying this connection to $d\varphi$, we define *the second fundamental form* of φ (see, for example, [3]). For any vector fields *X*, *Y* in *M*,

$$\nabla d\varphi(X, Y) = \nabla^\varphi_X\big(d\varphi(Y)\big) - d\varphi(\nabla^M_X Y). \tag{1}$$

Using the second fundamental form of a map φ, one can define the tension field of the map φ, as follows:

$$\tau(\varphi) = trace\, \nabla d\varphi.$$

Note that $\tau(\varphi)$ is a section on the pull-back bundle $\varphi^{-1}TN$.
At any point $x \in M$, define (see [3]) the vertical space of φ at x as $\mathrm{V}_x = \ker d\varphi_x$,and the horizontal space of φ at x as its orthogonal complement: $\mathcal{H}_x = \mathrm{V}_x^\perp$. For any point $x \in M$that is not a critical point (i.e. the inequality $rank\, d\varphi_x < \min\{m, n\}$ is not satisfied), the assignment $\mathcal{H}: x \to \mathcal{H}_x$ define a smooth distribution called the *horizontal distribution*, and the assignment $\mathrm{V}: x \to \mathrm{V}_x$ define the *vertical distribution*.
For each $x \in M$, $\varphi^{-1}(\varphi(x))$ is called *the fiber* through *x* and V_x, and for *x* that is not a critical point, gives the tangent space to the fibre through *x*.
Let *x ∈M* be any point. The map *φ* is called *horizontally conformal* at *x* (see [3]) if:

$$d\varphi_x: T_xM_{|_{\mathcal{H}_x \times \mathcal{H}_x}} \to T_{\varphi(x)}N$$

is a surjective map and there exists a positive function λ(x), called the *dilation* of *φ* at *x*, such that for any horizontal vector fields *X*, *Y* on *M*, the following equation holds true:

$$h\big(d\varphi_x(X), d\varphi_x(Y)\big) = \lambda^2(x) g(X, Y).$$

In is known (see [6], [3], [9]) that for a Riemannian manifold *(M^m, g)* and *S* a submanifold of *M*, for any point $x \in S$, with respect to the metric *g*, there is an orthogonal decomposition of the tangent bundle:

$$T_xM = T_xS \oplus T_xS^\perp.$$

Moreover,

$$\nabla^M_X Y = \nabla^S_X Y + A(X, Y), \tag{2}$$

for all vector fields *X*, *Y* on *S*, where ∇^M and ∇^K denote the Levi-Civita connections on *M* and *N*, respectively.
The map $A: TS \times TS \to TS^\perp$ is a bilinear one, called the *second fundamental form* of the submanifold *S*.
At each point *x* of the submanifold *S*, let us consider the normal vector to the submanifold (see [6], [3], [9]):

$$\mu^{V} = \frac{1}{q}\sum_{r=1}^{q} A(e_i, e_i), \tag{3}$$

where $\{e_1, e_2, \dots, e_q\}$ is an orthonormal basis for $T_x S$. This vector is called the *mean curvature* of the submanifold *S* in *x*.

We underline the fact that, when we consider inclusion maps of submanifolds or isometric immersions, the second fundamental form of a map, as defined in equation (1), coincides with the second fundamental form of a submanifold, as given in equation (2).

In what follows, we will consider the case where the submanifold *S* is a fiber of a smooth map $\varphi: M \to N$ between Riemannian manifolds.

Eells and Sampson (see [5]) proved that the harmonicity of a smooth map is equivalent to the vanishing of its tension field.

When introducing a vector field *V* on the source manifold of a smooth map between Riemannian manifolds, the notion of harmonicity can be extended, in a natural way, to *V*-harmonicity.

DEFINITION 2.1 (see [4], [10]) Let $\varphi: M \to N$ be a smooth map between two Riemannian manifolds (M^m, g) and (N^n, h), and let *V* be a smooth vector field on *M*.

φ is called *V-harmonic* if the *V-tension field of* φ vanishes, i.e.

$$\tau_V(\varphi) := \tau(\varphi) + d\varphi(V) = 0,$$

where $\tau(\varphi)$ is the tension field of the map φ.

P. Baird and J. C. Wood (see Proposition 4.5.3, [3]) found for the tension field of a smooth horizontally conformal submersion between Riemannian manifolds an expression using the mean curvature of its fibers and the gradient of its dilation. More precisely, if $\varphi: M^m \to N^n$ is a smooth horizontally conformal submersion with dilation $\lambda: M \to (0, \infty)$ and μ^V the mean curvature of its fibers, then:

$$\tau(\varphi) = -(n-2)d\varphi(grad \ln \lambda) - (m-n)d\varphi(\mu^V). \tag{4}$$

Using equation (4), Baird and Eells (see [2]) provided the following characterization of a harmonic map:

PROPOSITION 2.2 ([2]) Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and let $\varphi: M^m \to N^n$ be a non-constant horizontally conformal map with $\dim N > 3$ and dilation λ. Then, any two of the following statements imply the third:

1) φ is a harmonic map;
2) $grad\, \lambda$ is a vertical vector field;
3) the fibers of φ are minimal.

Zhao ([10]) found a similar expression forthe *V*-tension field of a smooth horizontally conformal submersion $\varphi: M^m \to N^n$ with a smooth vector field *V* on *M*:

$$\tau_V(\varphi) = d\varphi(V + \mathrm{grad}(\log \lambda^{2-n})) - (m-n)d\varphi(\mu^V). \tag{5}$$

A characterization similar to Proposition 2.2, was given in [10]:

PROPOSITION 2.3 ([10]) Let $\varphi: M^m \to N^n$ be a horizontally conformal map between Riemannian manifolds, with dilation $\lambda: M \to (0, \infty)$. Then, any two of the following assertions imply the third:

1) φ is a *V*-harmonic map;
2) $V + \mathrm{grad}(\log \lambda^{2-n})$ is vertical;
3) the fibers of φ are minimal.

3. GEOMETRIC CHARACTERIZATION OF V-HARMONIC MAPS

Considering a Riemannian manifold (M^m, g), let *S* be a submanifold on *M*, *V* a smooth vector field on *M*, and *A* the second fundamental form of the submanifold *S* in *M*. in [1] we define the notion of *V- minimality* of a submanifold.

DEFINITION 3.1 [1] The submanifold *S* is said to be *V-minimal* if trace (A) – V is a section of the tangent bundle *TS.*

When considering a differentiable map, and if the submanifold is exactly a fiber of the map, the *V*-minimality can be read/interpreted as: trace (A) – V is a vertical vector.

Using the notions of *V*-harmonicity and *V*-minimality, a result similar to Proposition 2.3 can be proved.

THEOREM 3.2 Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and let $\varphi: M^m \to N^n$ be a horizontally conformal submersion, where $n \geq 3$. Let V be a smooth vector field on M , and let $\lambda: M \to (0, \infty)$ denote the dilation of φ. Then, any two of the following statements imply the third:

1) φ is a *V*-harmonic map;
2) grad $(\log \lambda^{2-n})$ is a vertical vector field;
3) the fibres of φ are *V*-minimal.

Proof:

Let $x \in M$, $\mathrm{V}_x = \ker d\varphi_x$ denote the vertical space of φ at *x*, and $\mathcal{H}_x = \mathrm{V}_x^{\perp}$ the horizontal space of φ at *x*. Then, with respect to g, we have the orthogonal decomposition of the tangent vector bundle TM:

$$T_x M = \mathrm{V}_x \oplus \mathcal{H}_x,$$

where the vertical distribution has dimension $m - n$, and the horizontal distribution has dimension n.

Taking into account (1), (2) and (3), the mean curvature of the fiber is:

$$\mu^{\mathrm{V}} = \frac{1}{m-n}\, trace\ (\mathrm{A}) \Rightarrow trace\ (\mathrm{A}) = (m-n)\mu^{\mathrm{V}} \tag{6}$$

Using equation (6), the equality (5) can be written as follows:

$$\tau_V(\varphi) = d\varphi(V) + d\varphi(\mathrm{grad}(\log \lambda^{2-n})) - (m-n)d\varphi(\mu^{\mathrm{V}}) = \tag{7}$$

$$= d\varphi(V) - d\varphi(trace\ (\mathrm{A})) + d\varphi(\mathrm{grad}(\log \lambda^{2-n})) =$$

$$= d\varphi(V - trace\ (\mathrm{A})) + d\varphi(\mathrm{grad}(\log \lambda^{2-n})).$$

In (7), φ being *V*-harmonic is equivalent to $\tau_V(\varphi) = 0$, grad $(\log \lambda^{2-n})$ being a vertical vector field is equivalent to $d\varphi$ (grad $(\log \lambda^{2-n})) = 0$, and the fibers bein minimal is equivalent to $d\varphi(V - trace\ (\mathrm{A})) = 0$. Thus, the conclusion of the theorem follows.

Using the equation (5) a geometric characterization of *V*-harmonic maps, similar to the one found by Baird and Eells (see [2]), can be given.

THEOREM 3.3 Let us consider two Riemannian manifolds (M^m, g) and (N^n, h), a smooth vector field V on M, and $\varphi: M^m \to N^n$ a smooth, non-constant, horizontally conformal map, with $m, n \geq 1$. Let λ denote the dilation of φ. Then φ is V-harmonic if and only if the following equality holds:

$$\mathcal{H}(V) + \mathcal{H}(grad(\log \lambda^{2-n})) - trace(A) = 0,$$

where A denotes the second fundamental form of the fiber.

Proof: The *V*-harmonicity condition of φ (see equation (5)) translates in:

$$d\varphi(V) + d\varphi(\text{grad}(\log \lambda^{2-n})) - (m-n)d\varphi(\mu^{\text{V}}) = 0$$

which can be written as:

$$d\varphi[V + \text{grad}(\log \lambda^{2-n}) - (m-n)\mu^{\text{V}}] = 0 \text{ , or}$$

$$V + \text{grad}(\log \lambda^{2-n}) - (m-n)\mu^{\text{V}} \text{ is a vertical vector field.}$$

The verticality condition of the above vector, is equivalent tosaying that:

$$\mathcal{H}(V) + \mathcal{H}(\text{grad}(\log \lambda^{2-n})) - trace(A) = 0.$$

Acknowledgements: The author was partly supported by “Dunarea de Jos” University of Galati Grant, Project number RF 2488/31.05.2024.

References

1. Aprodu M. A., *V-minimal submanifolds*, preprint https://arxiv.org/pdf/2306.02104
2. Baird P., Eells J., A conservation law for harmonic maps, Lecture Notes in Mathematics, Springer Berlin, 894 (1981), 1–25.
3. Baird P., Wood J. C., *Harmonic Morphisms Between Riemannian Manifolds*, Clarendon Press Oxford, 2003.
4. Chen Q., Jost J., Wang G., *A Maximum Principle for Generalizations of Harmonic Maps in Hermitian, Affine, Weyl, and Finsler Geometry*, J. Geom. Anal., 25 (2015), 2407–2426.
5. Eells J., Sampson J. H., *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86 (1964), 109-160.
6. Ianuș S., *Geometrie diferențială cu aplicații în teoria relativității*, Editura Academiei Române, 1983.
7. Lee J. M., *Introduction to Smooth Manifolds*, Springer, Graduate Texts in Mathematics 218, 2013.
8. O'Neill B., *The fundamental equation of a submersion*, Michigan Math. J., 13 (4) (1966), 459-469.
9. Urakawa H., *Calculus of Variations and Harmonic Maps*, Transactions of Mathematical Monographs 132, American Mathematical Society, 1993.
10. Zhao G., *V-harmonic morphisms between Riemannian manifolds*, Proc. of the American Mathematical Society 148 (3) (2019) 1351-1361.