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Numerical method for approximating a function

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Abstract
In this paper we present the stages of implementing a numerical method to approximate a function. In order to achieve the objective of this work, we consider a function defined on an interval [a,b] and select three nodes from within the assumed interval. At these three chosen points, we will also know the values of the function at these points and also the values of the first order derivatives at the three points. An interpolation polynomial, of minimum degree, with the assumed nodes will be obtained.

Keywords: Hermite interpolation, Lagrange polynomials, error delimitation.

1. INTRODUCTION

We consider the function $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = xe^{-x^2} \tag{1}$$

We take three distinct nodes, x_0, x_1 and x_2, given in Table 1. In the considered points, the values were calculated $f(x_i), i = \overline{0,2}$ and $f'(x_i), i = \overline{0,2}$. The function f is differentiable at the considered points.

Table 1. Values used in the approximation of the function *f*

x	$x_0 = 0.5$	$x_1 = 1.2$	$x_2 = 2$
$f(x) = xe^{-x^2}$	0.38940	0.28431	0.03663
$f'(x) = (1-2x^2)e^{-x^2}$	0.38940	−0.44542	−0.12820

We propose identifying a polynomial of minimum degree, $P(x)$, that interpolates the function f. In this case, the sought polynomial will be of degree five. That satisfies the conditions [3]:

$$P(x_i) = f(x_i), P'(x_i) = f'(x_i), i = \overline{0,2} \tag{2}$$

Thus, we calculate the Hermite interpolation polynomial corresponding to the function f based on the values given in Table 1. In many applications, it is important to consider both the interpolation polynomial P, which approximates the function f, and its derivative P' that interpolates the derivative function f' [1].

2. RESULTS AND DISCUSSION

For the calculation of the Hermite interpolation polynomial, we apply theorem 3.9 from [2 page 136]. Using the points x_0, x_1 and x_2 from Table 1, together with the corresponding function values f in these nodes and the first derivative values, we determine the Lagrange polynomials using the formula 1.9 from [5 page 3]. For convenience in the calculation, we used the first five decimals. By calculating, we get:

$$L_0(x) = \frac{(x-1.2)(x-2)}{(0.5-1.2)(0.5-2)} = 0.95238 \cdot x^2 - 3.04761 \cdot x + 2.28571 \tag{3}$$

$$L_1(x) = \frac{(x-0.5)(x-2)}{(1.2-0.5)(1.2-2)} = -1.78571 \cdot x^2 + 4.46428 \cdot x - 1.78571 \tag{4}$$

$$L_2(x) = \frac{(x-0.5)(x-1.2)}{(2-0.5)(2-1.2)} = 0.83333 \cdot x^2 - 1.41666 \cdot x + 0.5 \tag{5}$$

Then, we derive the Lagrange polynomials obtained above and calculate the derivative values at the points considered in Table 1. It is obtained:

$$L'_0(x) = 1.90476 \cdot x - 3.04761 \Rightarrow L'_0(0.5) = -2.09523 \tag{6}$$

$$L'_1(x) = -3.57142 \cdot x + 4.46428 \Rightarrow L'_1(1.2) = 0.17857 \tag{7}$$

$$L'_2(x) = 1.66666 \cdot x - 1.41666 \Rightarrow L'_2(2) = 1.91666 \tag{8}$$

We continue to use formula 3.56 from [6 page 109] for the calculation of the Hermite interpolation polynomial.

$$u_0(x) = [1 - 2(x-0.5) \cdot (-2.09523)] \cdot (0.9523 \cdot x^2 - 3.04761 \cdot x + 2.28571)^2 \tag{9}$$

$$u_0(x) = 3.80083 \cdot x^5 - 25.31884 \cdot x^4 + 63.52255 \cdot x^3 - 73.32181 \cdot x^2 + 37.15156 \cdot x - 5.72199 \tag{10}$$

$$v_0(x) = (x-0.5) \cdot (0.9523 \cdot x^2 - 3.04761 \cdot x + 2.28571)^2 \tag{11}$$

$$v_0(x) = 0.90702 \cdot x^5 - 6.25847 \cdot x^4 + 16.54413 \cdot x^3 - 20.75272 \cdot x^2 + 12.19042 \cdot x - 2.61223 \tag{12}$$

$$u_1(x) = [1 - 2(x-1.2) \cdot 1.17857] \cdot (-1.78571 \cdot x^2 + 4.46428 \cdot x - 1.78571)^2 \tag{13}$$

$$u_1(x) = -1.13883 \cdot x^5 + 10.24951 \cdot x^4 - 32.17210 \cdot x^3 + 43.27576 \cdot x^2 - 23.91553 \cdot x + 4.55533 \tag{14}$$

$$v_1(x) = (x - 1.2) \cdot (-1.78571 \cdot x^2 + 4.46428 \cdot x - 1.78571)^2 \tag{15}$$

$$v_1(x) = 3.18876 \cdot x^5 - 19.77033 \cdot x^4 + 45.43990 \cdot x^3 - 47.51260 \cdot x^2 + 22.32134 \cdot x - 3.82651 \tag{16}$$

$$u_2(x) = [1 - 2(x - 2) \cdot 1.91666] \cdot (0.83333 \cdot x^2 - 1.41666 \cdot x + 0.5)^2 \tag{17}$$

$$u_2(x) = -2.66197 \cdot x^5 + 15.06918 \cdot x^4 - 31.35030 \cdot x^3 + 30{,}04593 \cdot x^2 - 13.23601 \cdot x + 2.16666 \tag{18}$$

$$v_2(x) = (x - 2) \cdot (0.83333 \cdot x^2 - 1.41666 \cdot x + 0.5)^2 \tag{19}$$

$$v_2(x) = 0.69443 \cdot x^5 - 3.74995 \cdot x^4 + 7.56243 \cdot x^3 - 7.09716 \cdot x^2 + 3.08332 \cdot x - 0.5 \tag{20}$$

If we replace the expressions obtained in (9)-(20) in formula 3.57 from [6 page 109], we obtain the Hermite interpolation polynomial of the function f for the considered points.

$$P(x) = -0.0974 \cdot x^5 + 0.45666 \cdot x^4 - 0.32659 \cdot x^3 - 1.15539 \cdot x^2 + 1{,}59187 \cdot x - 0.10236 \tag{21}$$

By deriving the polynomial of degree 5, $P'(x)$, the following expression is obtained:

$$P'(x) = -0.487 \cdot x^4 + 1.82664 \cdot x^3 - 0.97977 \cdot x^2 - 2.31078 \cdot x + 1{,}59187 \tag{22}$$

In order to highlight the fulfillment of the conditions given by (2), we graphically represented the obtained Hermite interpolation polynomial and P', together with the given values from Table 1 (Fig. 1, Fig. 2). It can be seen that the interpolation polynomial satisfies the conditions established by (2).

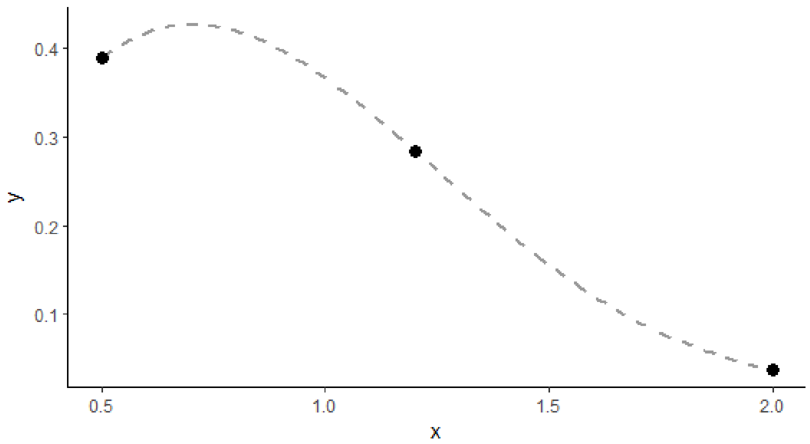

Fig. 1. Representation of the interpolation polynomial vs. initial values [7]

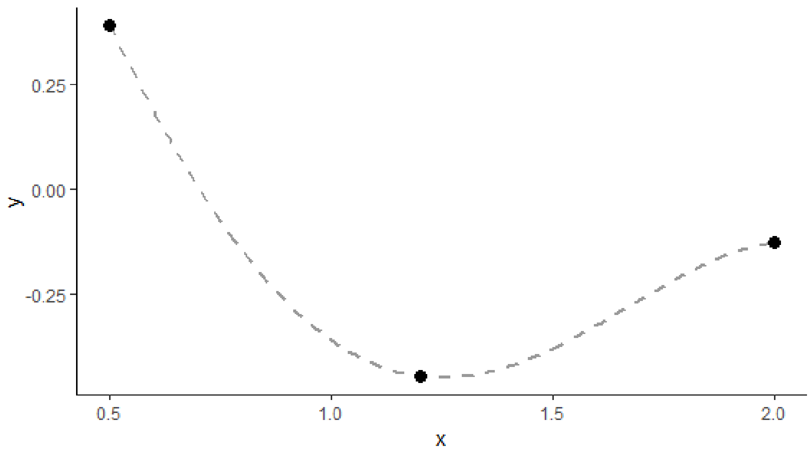

Fig. 2. Representation of the derived interpolation polynomial vs. initial values [7]

To show/demonstrate the efficiency of the interpolation method used, we graphically represented both the function $f(x) = xe^{-x^2}$ and the obtained interpolation polynomial P given by the expression (21) in Fig. 3. Therefore, we can see that on the interval [0.5,2], the graphic representations of the two functions overlap very well.

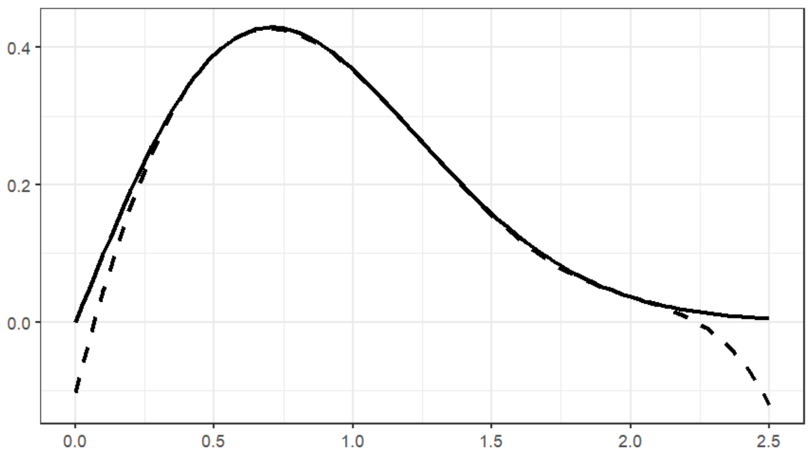

Fig. 3. The function $f(x) = xe^{-x^2}$ (solid curve) vs. the Hermite interpolation polynomial (dashed curve) [7]

To give a delimitation of the interpolation error we use theorem 2 from [4 page 311]. Thus for the distinct points given in Table 1, x_0, x_1 and x_2, $f \in C^6[0.5, 2]$ the error expression is:

$$(x-0.5)^2(x-1.2)^2(x-2)^2 \frac{f^{(6)}(\xi)}{6!}, \xi \in [0.5,2] \tag{23}$$

Fig. 4. represents the $xe^{-x^2} - P(x)$ error [3] in the presented interpolation method.

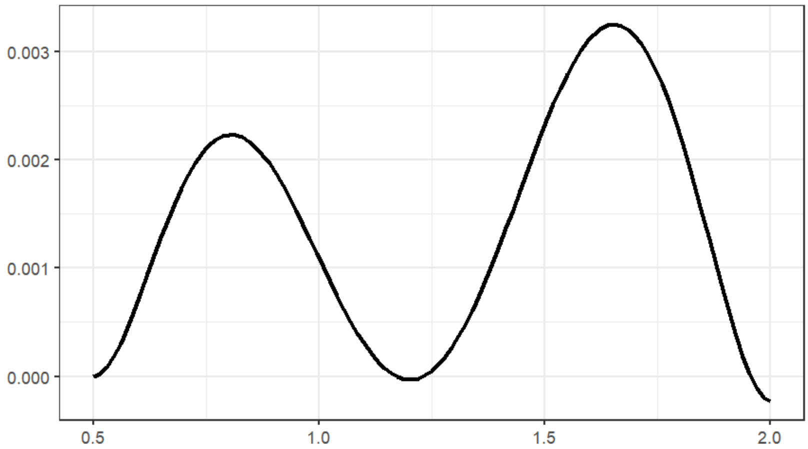

Fig. 4. Plot of the error in Hermite interpolation [7]

3. CONCLUSIONS

In this paper, we presented the Hermite interpolation method for the function $f(x) = xe^{-x^2}$ relative to three distinct points and applying the Lagrange form. The expressions of the interpolation polynomial and the derived interpolation polynomial were calculated. These polynomials were used to verify the imposed initial conditions. The error obtained in the interpolation of the f function was also given.

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