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Using rational functions to improve the results of approximating a function

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Abstract
In this article, we use rational functions in order to improve the results obtained in the approximation of a function in an interval [a, b]. For this we will approximate the function chosen as an example in this study using a ratio of two polynomials. To determine the two polynomials, we use the Taylor series expansion about the point x = 0 of the function chosen and the Padé approximation. Also, to highlight the accuracy of the obtained approximation, we analyze the absolute error between the initial function and the Pade approximation respectively and the Taylor series in the considered interval. The analyzed data, in the chosen interval, highlights much better results obtained by the Padé approximation compared to the Taylor series.

Keywords: Rational function, Taylor series, Padé approximations, maximum error.

1. INTRODUCTION

In [4] we used the real function

$$\mathrm{f}\colon \mathrm{R} \rightarrow \mathrm{R},\ \mathrm{f}(x) = \mathrm{x}\exp(-\mathrm{x}) \tag{1}$$

to measure the accuracy of this approximation using three distinct interpolation methods: Lagrange interpolation, Newton interpolation and respectively, Neville interpolation. In order to obtain the interpolation polynomial in the case of the three methods, we used five distinct nodes. Finally, three fourth degree polynomials were obtained and used to obtain the approximation of the function for a particular case [4].

In this paper we aim to identify a possible approximation for the function *f*, so that the difference between the function *f* and the found approximation is as small as possible in the considered interval [8]. Thus, we will approximate the function *f* on the interval [a, b] using rational functions. In conclusion, we will express an approximation for the function *f* as a ratio of two polynomials [9]:

$$R_{m,n}(x) = \frac{p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots + p_mx^m}{q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_nx^n} \tag{2}$$

The polynomial in the denominator is different from the zero polynomial [11].

Thus, we use the Padé approximation method together with the Taylor series expansion of the function f to be able to make the transition to rational functions [2]. The Taylor series expansion of the function $\mathrm{f}(x) = \mathrm{x}\exp(-x)$ for $\mathrm{x} = 0$ is given by [7]:

$$\mathrm{f}(x) = a_0 + a_1\mathrm{x} + a_2x^2 + a_3x^3 + \cdots \tag{3}$$

Thus, we can obtain [7]:

$$\mathrm{f}(x) = a_0 + a_1\mathrm{x} + a_2x^2 + a_3x^3 + \cdots = R_{m,n}(x) + \mathrm{O}(x^{m+n+1}) \tag{4}$$

Here $\mathrm{O}(x^{m+n+1})$ represents the power series $\sum_{k=m+n+1}^{\infty} a_k x^k$ [11].

2. RESULTS AND DISCUSSION

We consider the real function

$$\mathrm{f}\colon [0,3] \rightarrow \mathrm{R},\ \mathrm{f}(x) = \mathrm{x}\exp(-\mathrm{x}) \tag{5}$$

and the Taylor series expansion around $x_0 = 0$ (Maclaurin):

$$\mathrm{f}(x) = \mathrm{x} - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5 - \frac{1}{120}x^6 + \frac{1}{720}x^7 - \frac{1}{5040}x^8 + \mathrm{O}(x^9)$$

To calculate the coefficients of the polynomials in the numerator and denominator of relation (2) we used the formulas given by (1.6) and (1.7) from [1, page 3]. The free term in the denominator is equal to 1, $q_0 = 1$ [5]. For example, the first orders of the Padé approximation are presented below along with the associated Taylor polynomial. We can also observe from what is presented below that if the degree of the polynomial in the denominator is equal to zero, $\mathrm{n} = 0$, the Padé approximation for the function *f* is identical to the associated Taylor polynomial [10].

$$\begin{gathered} T_2(x) = \mathrm{x} - x^2 \\ R_{2,0}(x) = \frac{x - x^2}{1} \\ R_{1,1}(x) = \frac{x}{1 + x} \end{gathered} \tag{6}$$

$$\begin{gathered} T_3(x) = \mathrm{x} - x^2 + \frac{1}{2}x^3 \\ R_{3,0}(x) = \frac{2x - 2x^2 + x^3}{2} \\ R_{1,2}(x) = \frac{2x}{2 + 2x + x^2} \\ R_{2,1}(x) = \frac{2x - x^2}{2 + x} \end{gathered} \tag{7}$$

$$T_4(x) = \mathrm{x} - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 \tag{8}$$

$$R_{4,0}(x) = \frac{6x - 6x^2 + 3\,x^3 - x^4}{6}$$

$$R_{1,3}(x) = \frac{6x}{6 + 6x + 3x^2 + x^3}$$

$$R_{2,2}(x) = \frac{6x - 2x^2}{6 + 4x + x^2}$$

$$R_{3,1}(x) = \frac{6x - 4x^2 + x^3}{6 + 2x}$$

$$T_5(x) = \mathrm{x} - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$R_{5,0}(x) = \frac{24x - 24x^2 + 12\,x^3 - 4x^4 + x^5}{24}$$

$$R_{1,4}(x) = \frac{24x}{24 + 24x + 12x^2 + 4x^3 + x^4}$$

$$R_{2,3}(x) = \frac{24x - 6x^2}{24 + 18x + 6x^2 + x^3} \tag{9}$$

$$R_{3,2}(x) = \frac{12x - 6x^2 + x^3}{12 + 6x + x^2}$$

$$R_{4,1}(x) = \frac{24x - 18x^2 + 6x^3 - x^4}{24 + 6x}$$

$$T_6(x) = \mathrm{x} - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5 - \frac{1}{120}x^6$$

$$R_{6,0}(x) = \frac{120x - 120x^2 + 60\,x^3 - 20x^4 + 5x^5 - x^6}{120}$$

$$R_{1,5}(x) = \frac{120x}{120 + 120x + 60x^2 + 20x^3 + 5x^4 + x^5}$$

$$R_{2,4}(x) = \frac{120x - 24x^2}{120 + 96x + 36x^2 + 8x^3 + x^4} \tag{10}$$

$$R_{3,3}(x) = \frac{60x - 24x^2 + 3x^3}{60 + 36x + 9x^2 + x^3}$$

$$R_{4,2}(x) = \frac{60x - 36x^2 + 9x^3 - x^4}{60 + 24x + 3x^2}$$

$$R_{5,1}(x) = \frac{120x - 96x^2 + 36x^3 - 8x^4 + x^5}{120 + 24x}$$

$$T_7(x) = \mathrm{x} - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 - \frac{1}{6}x^4 + \frac{1}{24}x^5 - \frac{1}{120}x^6 + \frac{1}{720}x^7$$

$$\begin{aligned}
R_{7,0}(x) &= \frac{720x - 720x^2 + 360\,x^3 - 120x^4 + 30x^5 - 6x^6 + x^7}{720} \\
R_{1,6}(x) &= \frac{720x}{720 + 720x + 360x^2 + 120x^3 + 30x^4 + 6x^5 + x^6} \\
R_{2,5}(x) &= \frac{720x - 120x^2}{720 + 600x + 240x^2 + 60x^3 + 10x^4 + x^5} \\
R_{3,4}(x) &= \frac{360x - 120x^2 + 12x^3}{360 + 240x + 72x^2 + 12x^3 + x^4} \\
R_{4,3}(x) &= \frac{120x - 60x^2 + 12x^3 - x^4}{120 + 60x + 12x^2 + x^3} \\
R_{5,2}(x) &= \frac{360x - 240x^2 + 72x^3 - 12x^4 + x^5}{360 + 120x + 12x^2} \\
R_{6,1}(x) &= \frac{720x - 600x^2 + 240x^3 - 60x^4 + 10x^5 - x^6}{720 + 120x}
\end{aligned} \tag{11}$$

$$T_8(x) = \mathrm{x} - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5 - \frac{1}{120}x^6 + \frac{1}{720}x^7 - \frac{1}{5040}x^8$$

$$\begin{aligned}
R_{8,0}(x) &= \frac{5040x - 5040x^2 + 2520\,x^3 - 840x^4 + 210x^5 - 42x^6 + 7x^7 - x^8}{5040} \\
R_{1,7}(x) &= \frac{5040x}{5040 + 5040x + 2520x^2 + 840x^3 + 210x^4 + 42x^5 + 7x^6 + x^7} \\
R_{2,6}(x) &= \frac{5040x - 720x^2}{5040 + 4320x + 1800x^2 + 480x^3 + 90x^4 + 12x^5 + x^6} \\
R_{3,5}(x) &= \frac{2520x - 720x^2 + 60x^3}{2520 + 1800x + 600x^2 + 120x^3 + 15x^4 + x^5} \\
R_{4,4}(x) &= \frac{840x - 360x^2 + 60x^3 - 4x^4}{840 + 480x + 120x^2 + 16x^3 + x^4} \\
R_{5,3}(x) &= \frac{840x - 480x^2 + 120x^3 - 16x^4 + x^5}{840 + 360x + 60x^2 + 4x^3} \\
R_{6,2}(x) &= \frac{2520x - 1800x^2 + 600x^3 - 120x^4 + 15x^5 - x^6}{2520 + 720x + 60x^2} \\
R_{7,1}(x) &= \frac{5040x - 4320x^2 + 1800x^3 - 480x^4 + 90x^5 - 12x^6 + x^7}{5040 + 720x}
\end{aligned} \tag{12}$$

Good results in approximating the function $f(x) = x\exp(-x)$ in the analyzed interval, [0,3], were obtained in the case $m = n = 4$. Next, we will detail how we obtained the expression for the Padé approximation, $R_{4,4}(x)$, used in approximating the function *f* in the studied interval. $R_{4,4}(x)$ has the form:

$$R_{4,4}(x) = \frac{p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4}{1 + q_1x + q_2x^2 + q_3x^3 + q_4x^4} \tag{13}$$

We use the associated Taylor polynomial $T_8(x)$ to form the relationship [6]:

$$\mathrm{R}_{4,4}(\mathrm{x}) = \frac{P_4(\mathrm{x})}{\mathrm{Q}_4(\mathrm{x})} \Rightarrow \mathrm{P}_4(\mathrm{x}) = \mathrm{Q}_4(\mathrm{x})\mathrm{T}_8(\mathrm{x}) + \mathrm{O}(\mathrm{x}^9) \tag{14}$$

Where we get:

$$\begin{gathered}
p_0 = 0 \\
p_1 = 1 \\
p_2 = -1 + q_1 \\
p_3 = \frac{1}{2} - q_1 + q_2 \\
p_4 = -\frac{1}{6} + \frac{1}{2}q_1 - q_2 + q_3 \\
-\frac{1}{6}q_1 + \frac{1}{2}q_2 - q_3 + q_4 = -\frac{1}{24} \\
\frac{1}{24}q_1 - \frac{1}{6}q_2 + \frac{1}{2}q_3 - q_4 = \frac{1}{120} \\
-\frac{1}{120}q_1 + \frac{1}{24}q_2 - \frac{1}{6}q_3 + \frac{1}{2}q_4 = -\frac{1}{720} \\
\frac{1}{720}q_1 - \frac{1}{120}q_2 + \frac{1}{24}q_3 - \frac{1}{6}q_4 = \frac{1}{5040}
\end{gathered} \tag{15}$$

We will solve the system formed by the last four equations with the unknowns q_1, q_2, q_3 and q_4:

$$\begin{pmatrix} -\frac{1}{6} & \frac{1}{2} & -1 & 1 \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{2} & -1 \\ -\frac{1}{120} & \frac{1}{24} & -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{720} & -\frac{1}{120} & \frac{1}{24} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{24} \\ \frac{1}{120} \\ -\frac{1}{720} \\ \frac{1}{5040} \end{pmatrix} \tag{16}$$

With the solutions:

$$\begin{gathered}
p_0 = 0,\ p_1 = 1,\ p_2 = -\frac{3}{7},\ p_3 = \frac{1}{14},\ p_4 = -\frac{1}{210} \\
q_1 = \frac{4}{7},\ q_2 = \frac{1}{7},\ q_3 = \frac{2}{105},\ q_4 = \frac{1}{840}
\end{gathered}$$

Substituting into relation (13) we obtain:

$$R_{4,4}(x) = \frac{x - \frac{3}{7}x^2 + \frac{1}{14}x^3 - \frac{1}{210}x^4}{1 + \frac{4}{7}x + \frac{1}{7}x^2 + \frac{2}{105}x^3 + \frac{1}{840}x^4} \tag{17}$$

To study the results obtained in approximating the function $\mathrm{f}(x) = \mathrm{x}\exp(-x)$ by the associated Taylor polynomial $T_8(x)$ given by (12) and the Padé approximation given by (17) we used the data in Table 1. This comparative analysis is necessary to determine the best approximation of the function in the interval [0,3]. The criterion used to compare the two results was given by the calculation of the absolute error [3].

Table 1. Absolute error values for the associated Taylor polynomial, $\mathrm{T}_8(\mathrm{x})$ and the Padé approximation $\mathrm{R}_{4,4}(\mathrm{x})$ in the interval [0,3]

x	$\mathrm{f}(x)$	$T_8(x)$	$\lvert f(x) - T_8(x)\rvert$	$R_{44}(x)$	$\lvert f(x) - R_{44}(x)\rvert$
0	0	0	0	0	0
0.2	0.1637462	0.16374615	0.00000005	0.1637462	0
0.4	0.268128	0.26812801	0.00000001	0.268128	0
0.6	0.329287	0.32928675	0.00000025	0.329287	0
0.8	0.3594632	0.35946012	0.00000308	0.3594631	0.0000001
1	0.3678794	0.36785714	0.00002226	0.3678792	0.0000002
1.2	0.3614331	0.3613203	0.0001128	0.3614321	0.000001
1.4	0.3452357	0.34479314	0.00044256	0.3452325	0.0000032
1.6	0.3230344	0.32159081	0.00144359	0.3230257	0.0000087
1.8	0.297538	0.29345036	0.00408764	0.2975174	0.0000206
2	0.2706706	0.26031746	0.01035314	0.2706271	0.0000435
2.2	0.2437669	0.21980536	0.02396154	0.2436828	0.0000861
2.4	0.2177231	0.16624172	0.05148138	0.2175715	0.0001516
2.6	0.1931113	0.08919831	0.10391299	0.1928542	0.0002571
2.8	0.1702682	-0.02862182	0.19889002	0.1698539	0.0004143
3	0.1493612	-0.21428571	0.36364691	0.1487219	0.0006393

We can see from the presented data that much better results are recorded in the case of the Padé approximation for $\mathrm{m} = \mathrm{n} = 4$ at the expense of using the associated Taylor polynomial. An argument in this regard is represented by the maximum error measured in the interval [0,3] in the case of the two analyzed methods:

$$\begin{gathered} \lvert f(x) - T_8(x)\rvert \leq 0.36364691 \\ \lvert f(x) - R_{44}(x)\rvert \leq 0.0006393 \end{gathered} \tag{18}$$

The efficiency of the Padé approximation $R_{4,4}(x)$ that the (17) compares to the associated Taylor polynomial $T_8(x)$ that the (12) in approximating the function $\mathrm{f}(x) = \mathrm{x}\exp(-x)$ in the interval [0,3] is highlighted in Figure 1. To show how well the determined rational function approximates the function *f*, we have plotted the function f and the Padé approximation in the interval [2.75,3], see Figure 2.

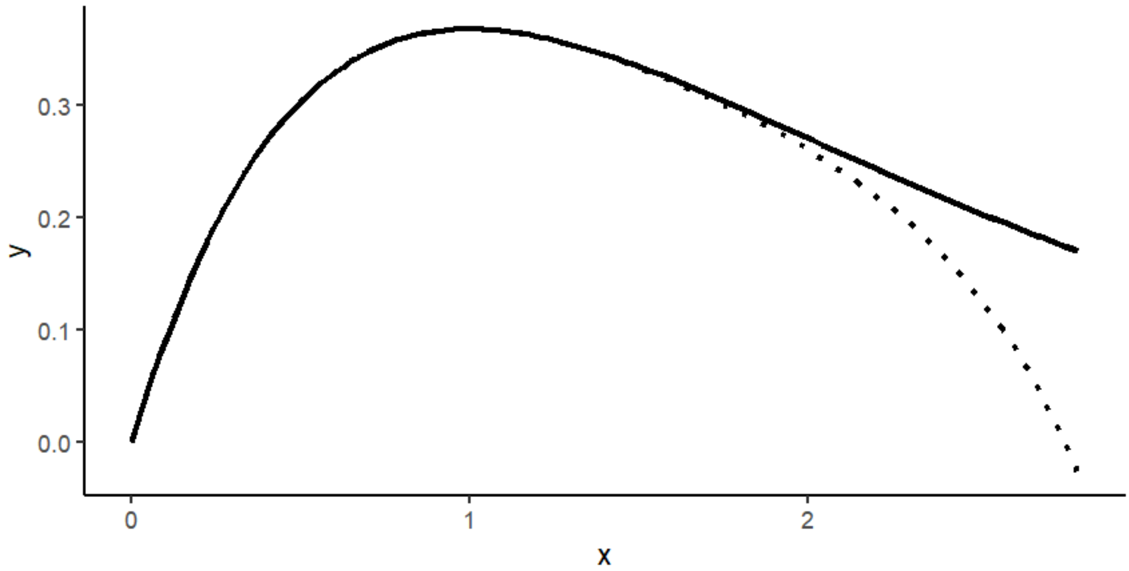


Fig. 1. The function graph $\mathrm{f}(x)=\mathrm{x}\exp(-x)$ *(solid curve) vs Padé approximation* $R_{4,4}(x)$ *(dashed curve) vs associated Taylor polynomial* $T_8(x)$ *(dotted curve) in the interval* $[0,3]$ *[12]*

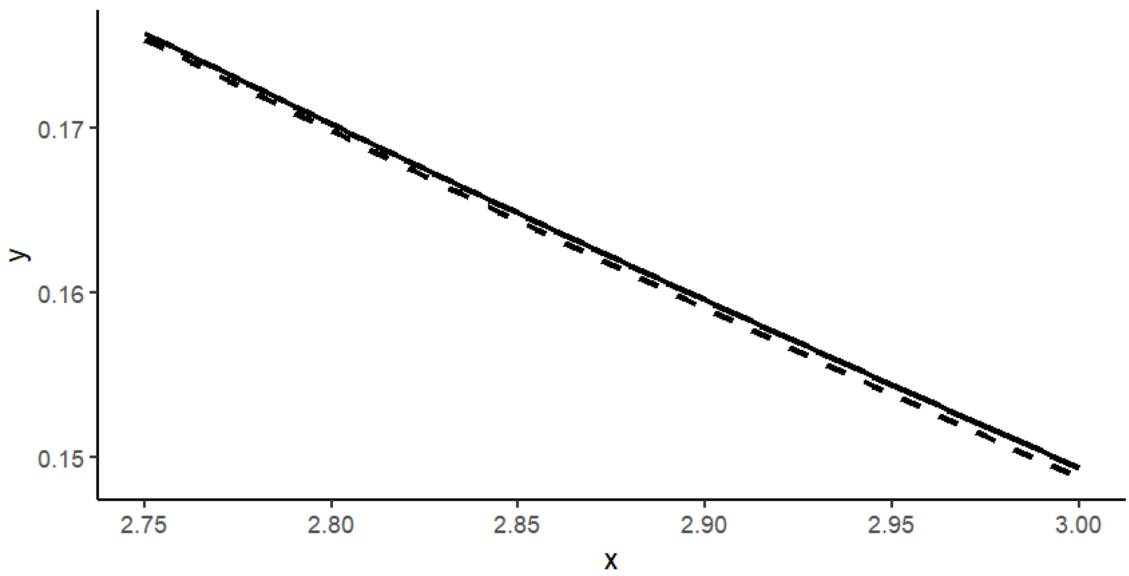
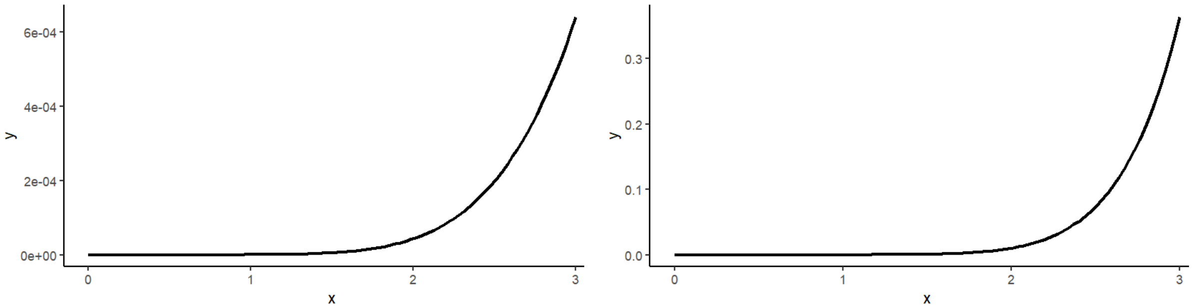


Fig. 2. The function graph $\mathrm{f}(x)=\mathrm{x}\exp(-x)$ *(solid curve) vs Padé approximation* $R_{4,4}(x)$ *(dashed curve) in the interval* $[2.75,3]$ *[12]*

The supremacy of the Padé approximation $R_{4,4}(x)$ over the associated Taylor polynomial is shown especially in the interval [0,0.6] where the error, $\mathrm{x}\exp(-x)-R_{4,4}(x)$, is equal to zero (see Table 1 and Figure 3). Also, the order of magnitude of the maximum absolute error value in the case of the Padé approximation is 0.1758% of the maximum absolute error value calculated in the case of the associated Taylor polynomial.



a) b)

Fig. 3. The graphic representation of absolute error $|\text{x}\exp(-x) - R_{4,4}(x)|$ *(a), vs the graphic representation of absolute error* $|\text{x}\exp(-x) - T_8(x)|$ *(b) in the interval* $[0,3]$ *[12]*

3. CONCLUSIONS

In this study, we tried to identify a method for approximating functions using the Taylor series expansion and rational functions. The results obtained in the interval [0, 3] for the two methods analyzed, see Table 1, highlight the efficiency of approximating the function f using the Padé approximation, $R_{4,4}(x)$. In choosing the degree of the polynomial used in the numerator and the polynomial in the denominator of the rational function, two aspects play a significant role: the accuracy of the desired approximation and the difficulty of the calculations performed. In the particular case analyzed in this study, we showed that the Padé approximation obtains much better results than the associated Taylor polynomial in approximating the function $\text{f}(x) = \text{x}\exp(-x)$ in the interval [0,3]. We also used the absolute error to highlight the efficiency of the approximation using rational functions.

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