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Implicitly defined V-harmonic maps

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Abstract
We extend the method of constructing harmonic morphisms via implicitly defined maps to the generalized context of V-harmonicity. We prove that if a map $F: N \times M \to Q$ satisfies specific holomorphic and V-harmonic conditions, the local solutions to the implicit equation defined by F are V-harmonic pseudo-horizontally homothetic (PHH) maps.

Keywords: Holomorphic maps, Harmonic maps, V-harmonic maps, Pseudo-horizontally homothetic maps, Riemannian manifold, Kähler manifold

1. PRELIMINARIES

The construction of harmonic morphisms through implicit maps was notably developed in previous works (see [2]), where it was also shown that level sets of maps satisfying certain geometric conditions--specifically, pseudo-horizontal homothety and harmonicity--are minimal submanifolds. Recently, the concept of harmonicity has been generalized to V-harmonicity by introducing a vector field V on the source manifold.

In this note, we combine the implicit function techniques for PHH maps (see [1],[2]) with the theory of V-harmonic maps (see [9]) to provide a new existence result for V-harmonic PHH submersions.

Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and V a smooth vector field on M.

Definition 1.1 ([3], [5]) We call a smooth map $\varphi: M \to N$ a *harmonic map* if it is a critical point of the energy functional: $E(\varphi) = \frac{1}{2}\int_D |d\varphi|^2 \; v_g$, for any compact domain $D \subset M$.

Remark: A harmonic map can be also characterized using the tension field which is defined (see [3]) as the divergence of $d\varphi$. Following Eells and Sampson (see [6]), a map is harmonic if and only if the tension field $\tau(\varphi) = 0$.

Definition 1.2 ([9]) A smooth map $\varphi: M \to N$ is called *V-harmonic* if its V-tension field vanishes:

$$\tau_V(\varphi) \coloneqq \tau(\varphi) + d\varphi(V) = 0,$$

where $\tau(\varphi)$ is the standard tension field of the map φ.

Remark: For a smooth map $\varphi: M \to N$, the tension field $\tau(\varphi)$, is a section on the pull-back bundle $\varphi^{-1}TN \to N$, and in local coordinates $(x_i)_{i=\overline{1,m}}$ on M and $(y_a)_{a=\overline{1,n}}$ on N respectively, it has the following expression (see for example [3], [5]):

$$\tau(\varphi) = \sum_{a=1}^{n} \tau(\varphi)^a \, \frac{\partial}{\partial y_a},$$

where the expression of $\tau(\varphi)^a, \forall a = 1, \ldots n$, is the following:

$$\tau(\varphi)^a = \sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 \varphi^a}{\partial x_i \partial x_j} - \sum_{k=1}^{m} {\Gamma_M}_{ij}^{k} \frac{\partial \varphi^a}{\partial x_k} + \sum_{b,c=1}^{n} {\Gamma_N}_{bc}^{a} \frac{\partial \varphi^b}{\partial x_i} \cdot \frac{\partial \varphi^c}{\partial x_j} \right).$$

By ${\Gamma_M}_{ij}^{k}$ and ${\Gamma_N}_{bc}^{a}$ are denoted the Christoffel symbols of M and N, and $\varphi^a = \varphi \circ y_a$.

If a smooth vector field V is considered on the manifold M, $V = \sum_{i=1}^{m} V_i \frac{\partial}{\partial x_i}$, the expresion of the V-tension field $\tau_V(\varphi)$ is the following:

$$\tau_v(\varphi) = \sum_{a=1}^{n} \tau(\varphi)^a \frac{\partial}{\partial y_a} + \sum_{a=1}^{n} \sum_{i=1}^{m} V_i \frac{\partial \varphi^a}{\partial x_i} \frac{\partial}{\partial y_a}.$$

Remark: 1. The notion of V-harmonic map naturally generalizes the classical notion of a harmonic map, but is not defined via a variational problem.

2. When V equals zero, or is a vertical vector, the two notions coincide.

3. In the case of a smooth function $f: M \to R$, the V-harmonicity reads:

$$\Delta_V(f) = \Delta(f) + \langle V, \nabla f \rangle = 0.$$

If (M^m, g) is a Riemannian manifold, (N^{2n}, J_N, h) is a Kähler manifold, and $\varphi: M \to N$ is a smooth map, where by ∇^M and ∇^N are denoted the Levi-Civita connections on M and N, respectively, and by $d\varphi_x^*: T_x M \to T_{\varphi(x)} N$ the adjoint map of the tangent linear map $d\varphi$, it is easy to see that $d\varphi^*(X)$ is a horizontal vector field on M, for X a section on $\varphi^{-1}TN$.

In this settings, the map φ is called *pseudo-horizontally (weakly) conformal at* $x \in M$, (see [8]) if and only if $[d\varphi_x \circ d\varphi_x^*, J_N] = 0$, and *pseudo-horizontally (weakly) conformal* if it is pseudo-horizontally (weakly) conformal for any point of the manifold M.

In local real coordinates $(x_i)_{i=1,\ldots,m}$ on M, and local complex coordinates $(z_a)_{a=1,\ldots,n}$ on N, and $\varphi^a = z_a \circ \varphi$ (see [8]), the local description of the *pseudo-horizontally (weakly) conformal (PHWC)* condition is:

$$\sum_{i,j=1}^{m} g^{ij} \frac{\partial \varphi^a}{\partial x_i} \cdot \frac{\partial \varphi^b}{\partial x_j} = 0, \forall a, b = 1, \ldots, n.$$

A special class of PHWC maps is the class of pseudo-horizontally homothetic maps.

Definition 1.3 ([2]) A smooth map $\varphi: M \to N$ is called *pseudo-horizontally homothetic* at $x \in M$, if it is PHWC and satisfies an extra condition at x: $\forall\, v \in T_x M$ a horizontal vector field and $\forall\, Y \in TN$ a vector field defined in a neighborhood of the point $\varphi(x)$,

$$d\varphi_x \left(\left(\nabla_v^M d\varphi^*(JY) \right)_x \right) = J_{\varphi(x)} d\varphi_x \left(\left(\nabla_v^M d\varphi^*(Y) \right)_x \right).$$

A map φ from a Riemannian manifold to a Kähler manifold is *pseudo-horizontally homothetic* (PHH) if it is pseudo-horizontally weakly conformal (PHWC) and satisfies an additional homothety condition on the horizontal distribution defined by the almost complex structure: for any horizontal vector field X on M and any vector field Y on N,

$$d\varphi \left(\left(\nabla_X^M d\varphi^*(JY) \right) \right) = J d\varphi \left(\left(\nabla_X^M d\varphi^*(Y) \right) \right).$$

Remark: The description of the PHH condition, using local normal coordinates for both manifolds M and

N, is the following (see [2]): consider $x_0 \in M$ a given point, $(x_i)_{i=1,\dots,m}$ localy defined normal real coordinates at x_0 such that all the vectors: $\frac{\partial}{\partial x_{2n+1}}(x_0), \dots, \frac{\partial}{\partial x_m}(x_0)$ are in $T_{x_0}M$, $(z_a)_{a=1,\dots,n}$ local normal complex coordinates at $\varphi(x_0) \in N$, and $\varphi^a = z_a \circ \varphi$,

$$\sum_{i=1}^{m} \frac{\partial \varphi^a}{\partial x_i}(x_0).\frac{\partial \varphi^b}{\partial x_i}(x_0) = 0$$

and

$$\sum_{j=1}^{m} \frac{\partial^2 \varphi^a}{\partial x_j \partial x_k}(x_0).\frac{\partial^2 \varphi^b}{\partial x_j}(x_0) = 0, \forall\, a,b = 1,\dots,n, \forall\, k = 1,\dots,2n.$$

2. IMPLICIT CONSTRUCTION OF V - HARMONIC MAPS

As we saw in the previous section, the notion of harmonicity can be naturally generalized to that of V - harmonicity, when a smooth vector field is given on the domain manifold. In this new context we can generalize the classical implicit function theorem for harmonic morphisms (Theorem 2.1, [1]).

Theorem 2.1: Let us consider three manifolds, (M^m, g) a Riemannian one, and (N^{2n}, J_N, h) and (Q^{2q}, J_Q, q) two Kähler manifolds. Let F be a smooth map defined by:

$$F: N \times M \to Q$$
$$(z,x) \to F(z,x),$$

such that:

1. the partial map $F_x: N \to Q$, for any $x \in M$, is holomorphic, and
2. for any $z \in N$, the partial map $F_z: M \to Q$, is a V- harmonic pseudo horizontally homothetic submersion.

Consider $\varphi: U \to N,\ U \subset Q \times M$ an open set, a smooth local solution for the equation:

$$F(\varphi(w,x),x) = w, \forall (w,x) \in U.$$

Then, for any $w \in Q$, the map $\varphi_w: M \to N$, defined by $\varphi_w(x) = \varphi(w,x)$ is a V- harmonic, pseudo-horizontally homothetic submersion.

Proof: The fact that φ_w is a PHH submersion depends only on the conformal structures and the PHH property of F_x and F_z. This was established in [1], and holds regardless of the tension field condition. We focus on proving the V −harmonicity.

Let us choose local real coordinates $(x_j)_{j=1,\dots,m}$ on M, and local complex coordinates $(z_i)_{i=1,\dots,n}$ and $(w_a)_{a=1,\dots,n}$ on N and Q, respectively.

We compute the partial differential, with respect to x_j, of the equation $F(\varphi(w,x),x) = w$:

$$\sum_{i=1}^{n} \frac{\partial F_x}{\partial z_i} \cdot \frac{\partial \varphi_w^i}{\partial x_j} + \sum_{i=1}^{n} \frac{\partial F_x}{\partial \bar{z}_i} \cdot \frac{\partial \bar{\varphi}_w^i}{\partial x_j} + \frac{\partial F}{\partial x_j} = 0.$$

As F_x is a holomorphic map the above equality reduces to:

$$\sum_{i=1}^{n} \frac{\partial F_x^a}{\partial z_i} \cdot \frac{\partial \varphi_w^i}{\partial x_j} + \frac{\partial F^a}{\partial x_j} = 0, \forall\, a = 1,\dots,n, \forall\, j = 1,\dots,m. \qquad (1)$$

To prove the $V-$ harmonicity condition for φ_w, we shall consider local normal coordinates.

Let $x_0 \in M$, $z_0 \in N$, and $w_0 \in Q$ fixed points such that $w_0 = F(z_0, x_0)$ and $(x_i)_{i=1,\dots,m}$ local normal real coordinates at x_0, $(z_j)_{j=1,\dots,n}$ local normal complex coordinates at z_0, $(w_j)_{j=1,\dots,n}$ local normal complex coordinates at w_0.

We differentiate equation (1) with respect to x_k:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial F_x^a}{\partial z_i} \cdot \frac{\partial \varphi_w^i}{\partial x_j}\right) + \frac{\partial^2 F^a}{\partial x_k \partial x_j} = 0,$$

which implies:

$$\sum_{i,r=1}^{n} \frac{\partial^2 F_x^a}{\partial z_r \partial z_i} \cdot \frac{\partial \varphi_w^r}{\partial x_k} \cdot \frac{\partial \varphi_w^i}{\partial x_j} + \sum_{i=1}^{n} \frac{\partial^2 F^a}{\partial x_k \partial z_i} \cdot \frac{\partial \varphi_w^i}{\partial x_j} + \sum_{i=1}^{n} \frac{\partial F_x^a}{\partial z_i} \cdot \frac{\partial^2 \varphi_w^i}{\partial x_k \partial x_j} + \frac{\partial^2 F^a}{\partial x_k \partial x_j} = 0,$$

$\forall\, a = 1, \dots, n, \forall j, k = 1, \dots, m.$

Multiplying the last equation with the matrix $(g^{ij})_{ij}$, we obtain:

$$\sum_{i,r=1}^{n} \sum_{k,j=1}^{m} \frac{\partial^2 F_x^a}{\partial z_r \partial z_i}\left(g^{kj} \frac{\partial \varphi_w^r}{\partial x_k} \cdot \frac{\partial \varphi_w^i}{\partial x_j}\right) + \sum_{i=1}^{n} \sum_{k,j=1}^{m} \frac{\partial^2 F^a}{\partial x_k \partial z_i} \cdot g^{kj} \cdot \frac{\partial \varphi_w^i}{\partial x_j}$$

$$+ \sum_{i=1}^{n} \sum_{k,j=1}^{m} \frac{\partial F_x^a}{\partial z_i} \cdot g^{kj} \cdot \frac{\partial^2 \varphi_w^i}{\partial x_k \partial x_j} + \sum_{k,j=1}^{m} g^{kj} \frac{\partial^2 F^a}{\partial x_k \partial x_j} = 0, \forall\, a = 1, \dots, n.$$

As $g^{kj}(x_0) = \delta^{kj}$ and φ_w is a PHWC map, the last equation reduces to:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial^2 F^a}{\partial x_j \partial z_i} \cdot \frac{\partial \varphi_w^i}{\partial x_j} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial F_x^a}{\partial z_i} \cdot \frac{\partial^2 \varphi_w^i}{\partial^2 x_j} + \sum_{j=1}^{m} \frac{\partial^2 F^a}{\partial^2 x_j} = 0, \forall\, a = 1, \dots, n. \tag{2}$$

The V – harmonicity condition of F_z: $\tau_V(F_z) = \tau(F_z) + dF_z(V) = 0$, is equivalent, in local coordinates, to:

$$\sum_{j=1}^{m} \frac{\partial^2 F_z^a}{\partial x_j^2} + \sum_{j=1}^{m} V_j \frac{\partial F_z^a}{\partial x_j} = 0, \forall a = 1, \dots, n \Rightarrow \sum_{j=1}^{m} \frac{\partial^2 F_z^a}{\partial x_j^2} = -\sum_{j=1}^{m} V_j \frac{\partial F_z^a}{\partial x_j}\ , \forall a = 1, \dots, n\, . \tag{3}$$

Substituting equation (3) in equation (2), we obtain:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial^2 F^a}{\partial x_j \partial z_i} \cdot \frac{\partial \varphi_w^i}{\partial x_j} = -\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial F_x^a}{\partial z_i} \cdot \frac{\partial^2 \varphi_w^i}{\partial^2 x_j} + \sum_{j=1}^{m} V_j \frac{\partial F_z^a}{\partial x_j}\ , \forall a = 1, \dots, n\, . \tag{4}$$

The V- harmonicity condition that φ_w must satisfy, reads, in local normal coordinates, as follows:

$$\sum_{j=1}^{m} \frac{\partial^2 \varphi_w^i}{\partial x_j^2} + \sum_{j=1}^{m} V_j \frac{\partial \varphi_w^i}{\partial x_j} = 0\ . \tag{5}$$

Multiplying this equation by the Jacobian of F_x, $\left(\frac{\partial F_x^a}{\partial z_i}\right)_{a=1,\dots,n}$, we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\partial F_x^a}{\partial z_i}\cdot\frac{\partial^2\varphi_w^i}{\partial x_j^2}+\sum_{i=1}^{n}\sum_{j=1}^{m}V_j\frac{\partial F_x^a}{\partial z_i}\cdot\frac{\partial\varphi_w^i}{\partial x_j}=0\,. \tag{6}$$

As proved in ([1], equation (2.4)), the following equality is true:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\partial^2 F^a}{\partial x_j\partial z_i}\cdot\frac{\partial\varphi_w^i}{\partial x_j}=0\,,$$

and comparing equations (4) and (6), the V-harmonicity condition for φ_w reduces to:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}V_j\frac{\partial F_x^a}{\partial z_i}\cdot\frac{\partial\varphi_w^i}{\partial x_j}+\sum_{j=1}^{m}V_j\frac{\partial F_z^a}{\partial x_j}=0\,,\forall a=1,\dots,n,$$

which is true from equation (1).

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