ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
MATHEMATICS, PHYSICS, THEORETICAL MECHANICS
FASCICLE II, YEAR XVII (XLVIII) 2025, No. 2
DOI: https://doi.org/10.35219/ann-ugal-math-phys-mec.2025.2.09

Analytical solutions in solving the parabolic partial differential equation using separation of variables and Fourier series

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Abstract
In this paper we present the analytical solution of a one-dimensional (1D) parabolic differential equation (PDE), encountered in many engineering studies under the name of ~~the~~ *heat equation*. To solve the equation, we use the method of separation of variables together with Fourier series. After applying the separation of variables, the initial equation decomposes into two ordinary differential equations (ODE). The equations obtained correspond to the spatial (x) and temporal (t) components. The spatial solution, for the first equation, is expressed by sinusoidal functions corresponding to the eigenfunctions, and the temporal component, for the second equation, has a characteristic exponential dependence. The determination of the Fourier series coefficients is based on the initial condition. Thus, we constructed an exact solution in the form of convergent series. The method used in solving the equation, together with the boundary conditions and initial conditions, can be used as a reference for validating numerical methods or practical experiments performed in the laboratory.

Keywords: parabolic differential equation, Fourier series.

1. INTRODUCTION

The parabolic, one-dimensional (1D) partial differential equations (PDEs) are used in the study of diffusion and transport phenomena. An example of a 1D parabolic equation is the heat equation [4]. To present the steps for solving this equation we consider the parabolic partial derivative equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4.2\cdot\ 10^{-6}}\frac{\partial u}{\partial t}, 0 < x < 1, t > 0, \tag{1}$$

with boundary conditions:

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t > 0, \tag{2}$$

and the initial condition given by:

$$u(x,1) = (1-x)e^{-5x}, 0 \le x \le 1. \tag{3}$$

Equation (1) can be considered to be similar to the homogeneous one-dimensional heat equation. From a physical point of view, it is considered a rectilinear bar, and we denote $u(x,t)$ the temperature at a point, located on the bar, arbitrarily chosen at time t. This equation has the form of equation (1-1) from [2, page 144], an equation often encountered in the study of physical phenomena. The equation is used to model the temperature distribution in a solid bar along one direction.

2. RESULTS AND DISCUSSION

To find the general solution of equation (1)-(3) we will use the method of separation of variables. In this sense, we will look for particular solutions for equation (1) - (3) of the form [2,4]:

$$u(x,t) = X(x) \cdot T(t) \tag{4}$$

We derive equation (4) and substituting into equation (1) we obtain:

$$X''(x)T(t) = \frac{1}{4.2 \cdot 10^{-6}} X(x)T'(t) \tag{5}$$

Equivalent relationship with:

$$\frac{X''(x)}{X(x)} = \frac{1}{4.2 \cdot 10^{-6}} \cdot \frac{T'(t)}{T(t)} = -k \tag{6}$$

We chose the minus sign in front of the positive constant k [5] (x, t are independent) because we want to find solutions for $X(x) \neq 0$. Also, we obtained relation (6) by excluding the solution $u(x,t) \equiv 0$. The equations are thus obtained:

$$\begin{gathered} X''(x) + kX(x) = 0 \\ T'(t) + 4.2 \cdot 10^{-6} \cdot kT(t) = 0 \\ X(0) = 0, X(1) = 0 \end{gathered} \tag{7}$$

The general solution of the first equation in (2) is given by [1, 5]:

$$X(x) = a\sin\left(x\sqrt{k}\right) + b\cos\left(x\sqrt{k}\right) \tag{8}$$

but, considering the boundary conditions [3], we obtain:

$$X(0) = 0 \Rightarrow b = 0\,, X(1) = 0 \Rightarrow a\sin\left(\sqrt{k}\right) = 0 \Rightarrow \sqrt{k} = n\pi \tag{9}$$

To simplify the calculation we can consider the scalar $a = 1$. Thus, we will obtain the solution:

$$X(x) = \sin(n\pi x), \ \ n = 1,2,3,\ldots \tag{10}$$

For the function $T(t)$, we obtain:

$$T(t) = e^{-4.2 \cdot 10^{-6} (n\pi)^2 t} \tag{11}$$

The general solution is given by [5]:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-4.2\cdot 10^{-6}(n\pi)^2 t} \sin(n\pi x) \tag{12}$$

where the Fourier coefficient, b_n, is determined by [5]:

$$u(x,0) = (1-x)e^{-5x} = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \tag{12}$$

The calculation steps for b_n are detailed below. Thus, using [4], we can write:

$$b_n = 2\int_0^1 (1-x)e^{-5x}\sin(n\pi x)dx$$
$$b_n = 2\left(\int_0^1 e^{-5x}\sin(n\pi x)dx - \int_0^1 xe^{-5x}\sin(n\pi x)dx\right) \tag{13}$$

Denoting the two integrals with I_1 and I_2 respectively, we can write the expression:

$$b_n = 2(I_1 - I_2) \tag{14}$$

To calculate the first integral we use [4] and obtain:

$$I_1 = \int_0^1 e^{-5x}\sin(n\pi x)dx = \frac{e^{-5x}}{25+(n\pi)^2}\left(-5\sin(n\pi x) - n\pi\cos(n\pi x)\right)\Big|_0^1$$
$$I_1 = \frac{n\pi[1+(-1)^{n+1}e^{-5}]}{25+(n\pi)^2} \tag{15}$$

To calculate the second integral, we use integration by parts and obtain:

$$I_2 = \int_0^1 xe^{-5x}\sin(n\pi x)dx = \frac{n\pi[10+(-1)^{n+1}e^{-5}((n\pi)^2+35)]}{(25+(n\pi)^2)^2} \tag{16}$$

Substituting expressions (15) and (16) into formula (14), we obtain the expression:

$$b_n = \frac{2n\pi[(n\pi)^2 + 15 - 10(-1)^{n+1}e^{-5}]}{(25+(n\pi)^2)^2} \tag{17}$$

In Fig. 1 we find the graphical representation of the solution of equation (1)-(3) for 20 spatial values, $t = 200$ and the first three terms of the Fourier series.

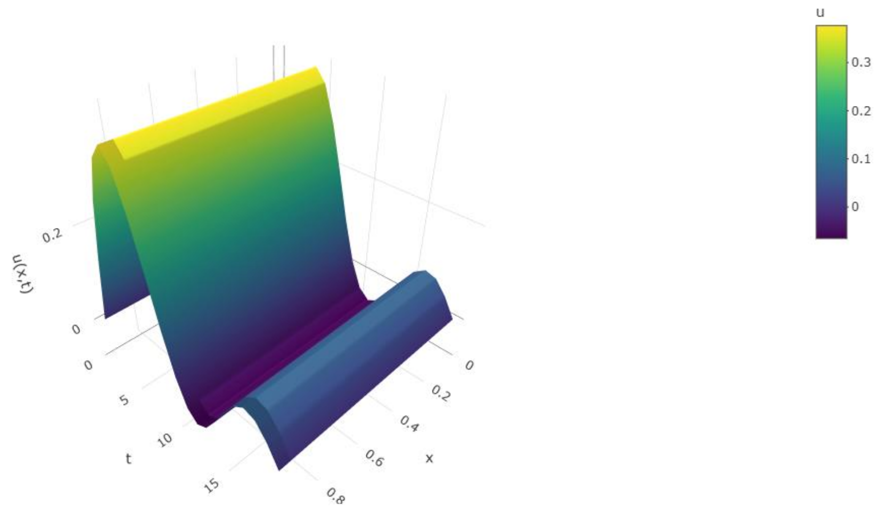

Fig. 1. Graphical representation for $u(x,t)$ versus x and t [6]

3. CONCLUSIONS

The numerical study presented for solving one-dimensional parabolic partial differential equations using the separation of variables and Fourier series method allowed for an in-depth detailing of how the initial temperature distributions evolve over time. Each eigenfunction contributes separately to the general solution, and its evolution is given by an exponential dependence. This aspect highlights how diffusion or transfer processes diminish over time. The analytical solution method provides exact solutions that can be used in the interpretation of physical phenomena, as well as for the validation of numerical methods.

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