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On units with complex Galois conjugates of equal absolute value

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Abstract
We investigate the following question:
Given a number field K with s real embeddings and 2t complex ones has a group of units $U \subset \mathcal{O}_K$ such that all elements in U have all its complex conjugates of same absolute value, does it follow that t = 1?
This fact has an interesting implication in complex hermitian geometry, namely it describes all Oeljeklaus–Toma manifolds carrying locally conformally Kähler structures. We prove that the stated question has an affirmative answer under a (relatively mild) condition on *K*, namely that for some finitely may extensions *L* of it, *L* has finitely many units lying on some specific circle.

Keywords: units, embeddings, l.c.K. metrics

1. INTRODUCTION

In view of Kronecker’s unit theorem, it is a natural question to look at algebraic units whose complex Galois conjugates have equal absolute value. This leads to the following question:

Give examples of number fields k of signature (s, t) (with $s,\ t > 0$) having subgroups U as above of (maximal) rank s.

The question arose from a problem in complex geometry, namely: such number fields would have given examples of compact complex manifolds (the so called "OT manifolds"), with interesting geometric structures, namely "l.c.K. metrics" (see Section 2.2 for the precise definitions).

Some geometrical facts hinted that such examples of number fields must necessarily have $t = 1$, but the proof of this came along a series of papers. First it was proven in [8] that one must have $t \leq s$; next, this result was widely extended in [3] where it was shown that the signature of any number field containing a unit whose complex conjugates are of same absolute value must obey a relation of the form $s = (2t + 2m)q - 2t$, for some integers $m \geq 0$ and $q \geq 2$. This left some cases (s, t) still open. Eventually, the remaining cases were settled in [1], hence proving that any field k as in the above question must indeed have $t = 1$.

2. PRELIMINARIES

2.1 NUMBER THEORY

Through this paper, k denotes a number field, that is, a finite field extension of $\mathbb{Q}$. Fix an algebraic closure $\overline{k}$. The primitive element theorem says that $k = \mathbb{Q}(\theta)$, for some $\theta \in k$, and let $f \in$

$\mathbb{Q}[X] \subset \overline{k}[X]$ be the minimal polynomial of θ. Let $\overline{\mathbb{Q}} = \overline{\mathbb{Q}}_{\mathbb{C}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Write $f = \prod_{\{i=1\}}^{n}(X - \theta_i)$, where $\theta_1 = \theta, \dots, \theta_n \in \overline{k}$ and $n = [k : \mathbb{Q}]$. According to Steinitz theorem, there is a $\mathbb{Q}$ - isomorphism of fields $\alpha : \overline{k} \to \overline{\mathbb{Q}}$ and let $a_i = \alpha(\theta_i)$, for any $i = \overline{1, n}$. For each such i, we have the field embedding $\sigma_i: k \to \overline{\mathbb{Q}} \subset \mathbb{C}$ given by $\theta \mapsto a_i$. The number of these embeddings is precisely n (as char$(k) = 0$). Moreover, these embeddings do not depend on the chosen primitive element, since if we pick another primitive element, say θ', its minimal polynomial would be f because of the automorphism of k given by $\theta \mapsto \theta'$. Also, any embedding $k \hookrightarrow \mathbb{C}$ must be one of the σ_i's because it must send θ into a root of $f \in \mathbb{Q}[Y] \subset \overline{\mathbb{Q}}[Y]$.

We rearrange these embeddings so that $\sigma_1, \dots, \sigma_s$ are all of them whose images are contained in $\mathbb{R}$, for some $s \geq 0$ (so these are the ones given by the real roots of $f \in \overline{\mathbb{Q}}[Y]$). As any complex root of f comes in pair with its conjugate, we may present the remaining embeddings as $\sigma_{\{s+1\}}, \overline{\sigma}_{\{s+1\}}, \dots, \sigma_{\{s+t\}}, \overline{\sigma}_{\{s+t\}}$, for some $t \geq 0$. In general, by the normal closure of k is meant the intermediate extension $k \subset \mathbb{Q}(\theta_1, \dots, \theta_n) := k^{nor} \subset \overline{k}$. Clearly, $\mathbb{Q} \subset k^{\text{nor}}$ is Galois (and hence $k \subset k^{\text{nor}}$ is so).

Definition 2.1.1. Let $\sigma_1, \dots, \sigma_s$ be the real embeddings of k and $\sigma_{s+1}, \overline{\sigma}_{s+1} := \sigma_{s+t+1}, \dots, \sigma_{s+t}, \overline{\sigma}_{s+t} := \sigma_{s+2t}$ be its complex embeddings. The signature of k is defined to be (s, t), denoted sgn(k). The group morphism $\ell: (k^*, \cdot) \to (\mathbb{R}^{s+t}, +)$

$$\ell(\alpha) = (\log|\sigma_1(\alpha)|, \dots, \log|\sigma_s(\alpha)|, 2\log|\sigma_{s+1}(\alpha)|, \dots, 2\log|\sigma_{s+t}(\alpha)|),$$

is called the logarithmic embedding of k.

Definition 2.1.2. The ring of integers of k is $\mathcal{O}_k = \{\alpha \in k;$ there is $f \in \mathbb{Z}[X]$ monic with $f(\alpha) = 0\}$, and the associated group of invertible elements, $\mathcal{O}_k^*$, is called the group of units of k. By $\mathcal{O}_k^{*,+}$ is denoted the subgroup of $\mathcal{O}_k^*$ of totally positive units, i.e. elements $\alpha \in \mathcal{O}_k^*$ for which $\sigma_i(\alpha) > 0$ for all $i = \overline{i, s}$, where $(s, t) = $ sgn(k).
As shown by Dirichlet, the image of $\mathcal{O}_k^*$ through ℓ is a full lattice inside a hyperplane of $\mathbb{R}^{s+t}$, in particular getting:

Theorem 2.1.3. [see e.g. [5], Chapter 5, Theorem 38] In the ring $\mathcal{O}_k$, there are units $\varepsilon_1, \dots, \varepsilon_r$, where $r = s + t - 1$ and $(s, t) = $ sgn(k), such that any unit $\varepsilon \in \mathcal{O}_k^*$ can be uniquely written as

$$\varepsilon = \zeta \varepsilon_1^{a_1} \dots \varepsilon_r^{a_r}$$

with $a_i \in \mathbb{Z}$ and $\zeta \in \mathcal{O}_k^*$ a root of unity.

Theorem 2.1.4 [see e.g. [5], Appendix 2, Theorem 1] For an extension of number fields K/k, any embedding of k extends to exactly [K: k] embeddings of K.

Definition 2.1.5. A subgroup U of $\mathcal{O}_k^{*,+}$ is called admissible if rk$_{\mathbb{Z}}(U) = $ s and $\pi(\ell(U))$ is a full lattice in $\mathbb{R}^s$, where π denotes the projection $\mathbb{R}^{s+t} \to \mathbb{R}^s$.

Remark 2.1.6. The following remarks will be useful in the following parts:
- the subgroup of totally positive units, $\mathcal{O}_k^{*,+}$ is of finite index in $\mathcal{O}_k^*$ (as the square of a unit has positive real images);
- for any number k field with $s, t \geq 1$ there exist admissible subgroups [6].

Definition 2.1.7. Let $R \in \overline{\mathbb{Q}}$, $R > 0$, be an algebraic integer and let k be identified with the image of a complex embedding of it. We say that k is R-finite if $|\mathcal{O}_K^* \cap \mathcal{C}(0, R)| < \infty$, where $K = k^{nor}(R)$ and $\mathcal{C}(0, R)$ is the circle centered in the origin and having radius R.

Remark 2.1.8. If R is also a unit, we have a natural structure of abelian group on $\mathcal{O}_k^* \cap \mathcal{C}(0,R)$, for any number field k containing R, given by

$$\alpha * \beta := \frac{\alpha\beta}{R},$$

for any $\alpha, \beta \in \mathcal{O}_k^* \cap \mathcal{C}(0,R)$. Plainly, $*$ is associative and commutative. The identity element is R and the inverse of α is $\overline{\alpha}$.

2.2 OELJEKLAUS – TOMA MANIFOLDS

These manifolds were introduced by K.Oeljeklaus and M. Toma in [6] as a generalization to higher dimensions of the Inoue surfaces S_M in [4]. Very briefly, their construction goes as follows. Begin with a number field K of signature $(s,t) > 0$ (cf. Section 2.1 for what follows). Letting $\mathbb{H} := \{z \in \mathbb{C}; \mathcal{I}m(z) > 0\}$, we see there are natural actions of $(\mathcal{O}_k, +)$ and respectively of $\mathcal{O}_k^{*,+}$ on $\mathbb{H}^s \times \mathbb{C}^t \subset \mathbb{C}^{s+t}$ by

$$a \cdot (z_1, \dots, z_{s+t}) \mapsto (z_1 + \sigma_1(a), \dots, z_{s+t} + \sigma_{s+t}(a))$$

and respectively

$$u \cdot (z_1, \dots, z_{s+t}) \mapsto (\sigma_1(u)z_1, \dots, \sigma_{s+t}(u)z_{s+t}).$$

The combined resulting action of $\mathcal{O}_K^{*,+} \ltimes \mathcal{O}_K$ is however not discrete in general. Still, in [6] it is shown that one can always find admissible subgroups $U \subset \mathcal{O}_K^{*,+}$ such that the action of $U \ltimes \mathcal{O}_K$ is discrete and cocompact: the resulting compact complex manifold is denoted $X(\mathcal{O}_K, U)$ and is called an *Olejeklaus-Toma manifold* (OT, for short).

As these manifolds do not admit Kähler metrics (cf. [6], Prop. 2.5) it is natural to ask whether other natural metrics (do) exist on them. One of the most interesting candidates are the *locally conformally Kähler* metrics (l.c.K, for short): these are those Hermitian metrics whose associated $(1,1) -$ forms ω have the property

$$d\omega = \theta \wedge \omega \quad (1)$$

for some closed $1 -$form θ (for more details see [2] or [7] for a more detailed account). The existence of such metrics on OT manifolds $X(K,U)$ can be read off the Galois properties of the group of units U. More precisely, it was shown that:

Proposition 2.2.1. (cf [3], Appendix by L. Battisti) An Oeljeklaus-Toma manifold $X(K,U)$ admits an l.c.K. metric if and only if for any unit $u \in U$ one has

$$|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)| \quad (2)$$

Recall also the following

Definition 2.2.2. (cf [6], Def. 1.5) An Oeljeklaus-Toma manifold $X(K,U)$ is called of simple type if there is no proper subfield $L \subset K$ such that $U \subset L$.

Definition 2.2.3. For a given Oeljeklaus-Toma manifold $X(K,U)$, a unit $u \in U$ will be called homothetical (resp. isometrical) if

$$|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)| > 1 \text{ (resp. } |\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)| = 1).$$

Notice that since $X(K,U)$ is non- Kähler, at least one of the units must be homothetical.

3. THE MAIN RESULTS

3.1. DISTINGUISHED GROUPS OF UNITS IN NUMBER FIELDS ASSOCIATED TO LCK MANIFOLDS

First we give a new, simpler proof of the result in [8]:

Proposition 3.1. Let $X(K,U)$ be an OT manifold. If $s<t$ and $X(K,U)$ is of simple type, then all units in K must be isometrical, getting in this way a contradiction with the existence of l.c.K. metrics on $X(K,U)$.

Proof. Since $\sigma_{s+1}(u)\overline{\sigma_{s+1}(u)}=\cdots=\sigma_{s+t}(u)\overline{\sigma_{s+1}(u)}=R^2$, if we take any $\sigma\in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, we get

$$\sigma(z_1)\sigma(\overline{z}_1)=\cdots=\sigma(z_t)\sigma(\overline{z}_t)=\sigma(R^2)$$

where $z_i:=\sigma_{s+i}(u)$, for any $i=\overline{1,t}$. As $s<t$, there must be at least one product of the form z_iz_j in the above equations, according to the box principle. We notice that not all of these products have just one real conjugate $\sigma(z_i)$, for otherwise, by taking absolute values, we get $r_iR=r_jR$ for at least two indices i,j, where $r_i:=\sigma_i(u)$, for $i=\overline{1,s}$, contradicting $\deg_{\mathbb{Q}}(u)=[K:\mathbb{Q}]$. Thus, we must have $r_ir_j=\sigma(z_i)\sigma(\overline{z}_i)$. But then

$$\sigma(R^2)=\sigma(z_i)\sigma(\overline{z}_i)=r_ir_j=|r_ir_j|=|\,\sigma(z_i)\sigma(\overline{z}_i)|=R^2.$$

As σ was arbitrary, we conclude that $R^2\in\mathbb{Q}$, and since R is a positive algebraic unit, we must have $R=1$.

Theorem 3.2. Let K be a number field having signature (s,t), with $s\geq 1$ and $t\geq 1$. Assume that U is a rank s subgroup of $\mathcal{O}_K^*$ such that for each $u\in U$ one has $|\sigma_{s+1}(u)|=|\sigma_{s+j}(u)|=R_u$, for all i,j. If there is a $\mathbb{Z}$ - basis $\{u_1,\dots,u_s\}$ of U such that K is R_{u_i}-finite for all i, then $t=1$. In particular, it follows that there is no OT manifold $X(K,U)$ of simple type carrying an l.c.K. metric.

Proof. Assume $t>1$. Consider

$$G_U:=\{u\in U|\sigma_{s+1}(u)=\cdots=\sigma_{s+t}(u)\}.$$

We show that $G_U\subset U$ is of finite index. For this, it suffices to show that $u_i^n\in G_U$, for any i. According to the hypothesis and Remark 2.1.8, $\mathcal{O}_{L_i}^*\cap\mathcal{C}(0,R_{u_i})$ is a finite abelian group, where $L_i=K^{nor}(R_{u_i})$, so there is $n\in\mathbb{N}$ such that $\alpha^{*n}=R_{u_i}$ for all $\alpha\in\mathcal{O}_{L_i}^*$. From the definition of the group law $*$ it follows immediately that $\alpha^n=R^n$. But the hypothesis on U says that $\sigma_{s+j}(u_i)\in\mathcal{O}_{L_i}^*\cap\mathcal{C}(0,R)$ for any $j=\overline{1,t}$. Therefore $u_i^n\in G_U$.

In particular, $rk_{\mathbb{Z}}(G_U)=s$. Consider the field $k:=\{x\in K|\sigma_{s+1}(x)=\cdots=\sigma_{s+t}(x)\}$ and denote by (s',t') its signature. As at least t embeddings of K lie above a single embedding of k, we see that, according to Theorem 2.1.4,

$$[K:k]\geq t \tag{3}$$

As all the complex embeddings of K lie above a single embedding τ of k, and hence all real embbedings of k distinct from τ lift to real embeddings of K, we must have

$$t'\in\{0,1\} \tag{4}$$

and also

$$s' - 1 \leq \frac{s}{[K:k]} \leq \frac{s}{t} \qquad (5)$$

(also by Theorem 2.1.4). More, from $G_U \subset O_K^*$ and Dirichlet's unit theorem (Theorem 2.1.3) we get

$$s' + t' - 1 \geq s.$$

Now, if $t' = 1$, from (3) we get $s' + 2t' \leq \frac{s}{t} + 2$, that is, $s' \leq \frac{s}{t}$; but (5) implies $s' \geq s$ hence we get $s \leq \frac{s}{t}$ which forces $t = 1$ since $s \geq 1$. If $t' = 0$, from (5) we get $s' - 1 \geq s$, so from (4) we get $\frac{s}{t} \geq s$ which forces again $t = 1$.

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