THE ANNALS OF “DUNAREA DE JOS” UNIVERSITY OF GALATI.
FASCICLE IX. METALLURGY AND MATERIALS SCIENCE
N^0. 4 – 2011, ISSN 1453 – 083X

A TRIBOLOGICAL MODEL FOR CONTINUOUS CASTING EQUIPMENT OF THE STEEL

Viorel MUNTEANU[1], Constantin BENDREA[2]
[1]Faculty of Metallurgy and Materials Science, Dunarea de Jos University of Galati
[2]Faculty of Science, Department of Mathematics, Dunarea de Jos University of Galati
email: munteanu_viorel48@yahoo.com

ABSTRACT

This paper presents a coupled thermo-elastic contact problem with tribological processes on the contact interface (friction, wear or damage). The unilateral contact between the cilindrical roll system and a deformable foundation (slab, bloom, etc) is modeled by the Kuhn-Tucker (normal compliance) conditions, involving damage and/or wear effect of contact surfaces. The continuum tribological model is based on gradient theory of the damage variable for studying crack initiation in fretting fatigue [11], [14], [15] , and the wear is described by Archard' s law. The friction law that we consider is a regularization of the Coulomb law.

The weak formulation of the quasistatic boundary value problem is described by using the variational principle of virtual power, the principles of thermodynamics and variational inequalities theory. Thus, the main results of existence for weak solution are established using a discretization method (FEM) and a fixed-point strategy [5].

KEYWORDS: Continuous casting, Thermoelastic contact, Friction, Wear, Fretting fatigue, Variational Inequalities, Galerkin Discretization Method

The elastic thermo-deformable body (walls of the mould, cilindrical rolls system) occupies a regular domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with surface Γ that is portioned into three disjoint measurable part $\Gamma = \Gamma_u \cup \Gamma_\sigma \cup \Gamma_c$ such that means $(\Gamma_u) > 0$. Let $[0,T]$ be the time interval of interest with $T > 0$. The body is clamped on $(0,T) \times \Gamma_u$ and therefore the displacement field vanishes there. We denote by S_d the spaces of second order symmetric tensors, while “$\cdot$” and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d or $\mathbb{R}^d$. Let n denote the unit outer normal on Γ, and everywhere in the sequel the index i, j run from 1 to d (summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable).

We also use the following notation and physical nomenclatures:

$\Omega_T = (0,T) x\, \Omega$;
$\overline{\Omega}_T = [0,T] \times (\Omega \cup \Gamma)$;
$\Gamma = \partial\Omega$; $\Gamma_T = (0,T) x\, \Gamma$;
$\Gamma_{iT} = (0,T) x\, \Gamma_i$, $i \in \{u\ ;\ \sigma\ ;\ c\}$;

$t \in [0,T]$ time variable;
$x \in \Omega$ spatial variable;
$u : \overline{\Omega}_T \to \mathbb{R}^d$ displacement vectorial field;
$\dot{u} = \left(\frac{\partial u_i}{\partial t}\right)$; $\ddot{u} = \left(\frac{\partial^2 u_i}{\partial t^2}\right)$ velocity and inertial vectorial fields, respectively;
$\sigma : \Omega_T \to S_d$ stress tensor field (second order Piola –Kirchhoff);
$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ strain tensor field (linearized tensor Green-St. Venant);
$\theta : \Omega_T \to \mathbb{R}$ temperature scalar field;

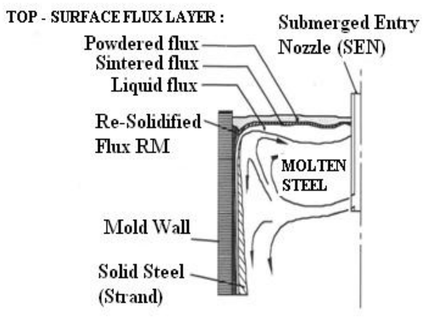

***Fig. 1a.** Schematic unilateral contact with pure sliping between the walls of mold and molten steel [5], [12], [14].*

THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI.
FASCICLE IX. METALLURGY AND MATERIALS SCIENCE
N^0. 4 – 2011, ISSN 1453 – 083X

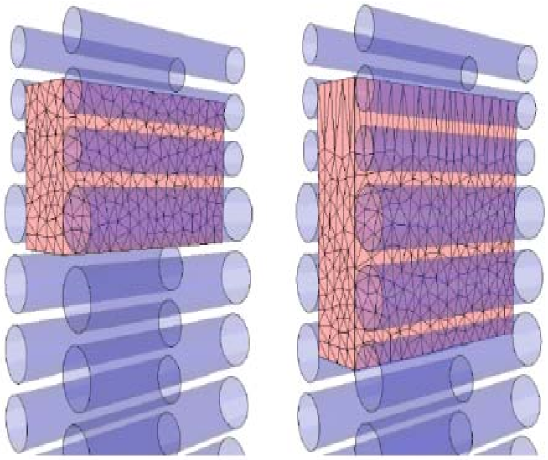

Fig. 1b. *Schematic unilateral contact with pure rolling between the support-rolls and steel slab [5], [12].*

We assume that a quasistatic process is valid and the constitutive relationship of an elastic-viscoplastic material can be written as

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{G}\varepsilon(u) - \mathcal{C}\theta \qquad (1.1)$$

where $\mathcal{A}$ and $\mathcal{G}$ are *nonlinear* operators whitch will be described below,

and $\mathcal{C} = (c_{ij})$ represents the *thermal expansion tensor*.

Here and below, in order to simplify the notation, we usually do not indicate explicitly the dependence of the functions on the variables $x \in \Omega$ (on the time $t \in [0,T]$ sometimes). Examples of constitutive laws of the form (2.1) can be constructed by using *thermal* aspects and *rheological* arguments, (see e.g. [3], [5], [6], [10], [11]) .

When $\mathcal{C} = O_d$ the constitutive law (1.1) reduces to the *Kelvin-Voigt viscoelastic* behaviour of the materials ,

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{G}\varepsilon(u), \qquad in\ \Omega_T \qquad (1.2)$$

Finally, the evolution of the temperature field is governed by the *heat transfer equation* (see [1], [3], [6]),

$$\dot{\theta} - div(K\nabla\theta) = q - \mathcal{C}\cdot\nabla\dot{u}, \quad \text{in } \Omega_T \qquad (1.3)$$

where : $K = (k_{ij})_{i,j=\overline{1,d}}$ is thermal conductivity tensor;

$\mathcal{C} = (c_{ij})_{i,j=\overline{1,d}}$ is thermal expansion tensor;

q represent the density of volume heat sources.

In order to simplify the description of the problem, a *homogeneous condition* for the temperature field is considered on $\Gamma_u \cup \Gamma_\sigma$,

$$\theta = \theta_r, \quad \text{on} \quad \Gamma_{uT} \cup \Gamma_{\sigma T} \qquad (1.4)$$

It is straightforward to extend the results shown in this paper to more general cases.

Also, we assume the associated *temperature boundary condition* is described on Γ_c ,

$$(K\nabla\theta)\cdot n = -k_e(\theta - \theta_r), \text{ on } \Gamma_{cT} \quad (1.5)$$

where θ_r is the reference temperature of the obstacle, and k_e is the heat exchange coefficient between the body and the rigid foundation.

Thus, the *thermo-mechanical problem* in the *clasic vectorial formulation*, can be written as follows :

Problem (P):

Find a displacement field $u : \overline{\Omega}_T \to \mathbb{R}^d$

a stress tensor field $\sigma : \overline{\Omega}_T \to \mathcal{S}_d$ and,

a temperature field $\theta : \overline{\Omega}_T \to \mathbb{R}$ such that,

$$\sigma = \mathcal{A}(\varepsilon(\dot{u})) + \mathcal{G}(\varepsilon(u)) - \mathcal{C}\cdot\theta \qquad (1.6)$$

$$-div\,\sigma(\varepsilon(u)) = f, \qquad in\ \Omega_T \qquad (1.7)$$

$$\dot{\theta} - div\,(K\nabla_x\theta) = q - \mathcal{C}\cdot\nabla_x\dot{u} \qquad (1.8)$$

$$u = 0, \qquad on\ \Gamma_{uT} \qquad (1.9)$$

$$\sigma(\varepsilon(u))\cdot n = g, \qquad on\ \Gamma_{\sigma T} \qquad (1.10)$$

$$\left.\begin{array}{l} u_n \le 0,\ \sigma_n \le 0 \ \text{ a.i.} \\ \quad u_n\cdot\sigma_n(u) = 0, \quad on\ \Gamma_{cT} \end{array}\right. \qquad (1.11)$$

$$|\sigma_\tau(u)| \le \mu|\mathcal{T}\sigma_n(u)|, \quad on\ \Gamma_{cT}\ i.e. \qquad (1.12)$$

$$|\sigma_\tau(u)| < \mu|\mathcal{T}\sigma_n(u)| \Rightarrow \dot{u}_\tau = 0 \qquad (1.13)$$

$$\left.\begin{array}{l} |\sigma_\tau(u)| = \mu|\mathcal{T}\sigma_n(u)| \Rightarrow \\ \quad (\exists)\lambda \ge 0 \ :\ \sigma_\tau(u) = -\lambda\dot{u}_\tau \end{array}\right. \qquad (1.14)$$

$$(K\nabla_x\theta)\cdot n = -k_e(\theta - \theta_r), \quad on\ \Gamma_{cT} \qquad (1.15)$$

$$\theta = \theta_r \quad , \qquad on\ \Gamma_{uT} \cup \Gamma_{\sigma T} \qquad (1.16)$$

$$u(0) = 0\ ;\ \theta(0) = \theta_0\ , \qquad in\ \Omega_T \qquad (1.17)$$

Here, $u_0 = 0$ and θ_0 represent the initial displacement and the initial temperature, respectively. Also, a volume force of density f acts in Ω_T and a surface traction of density g acts on $\Gamma_{\sigma T}$.

We will consider a nonlocal Coulomb friction law and in fact a regularization of it in order that the boundary terms in the formulation of our problem make sense. In the sequel, $\mathcal{T} : H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$ will represent a smoothing operator that is linear,

THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI.
FASCICLE IX. METALLURGY AND MATERIALS SCIENCE
N^0. 4 – 2011, ISSN 1453 – 083X

ontinuous and this satisfy,

$(\mathcal{T}\sigma(u))_n \leq 0 \; if \; \sigma_n(u) \leq 0$, $on \; \Gamma_{c|T}$ (1.18)

2.Variational formulation. Existence and uniqueness results

In order to obtain the *variational formulation* of *Problem* **(P)** , let us introduce additional notation and assumptions on the problem data.

Let, $\varepsilon : \mathbb{H}_1 \to \mathcal{H}_1$ and $div : \mathcal{H}_1 \to \mathbb{H}$ the *Hooke deformation* and *divergente operators*, respectively, defined by:

$$\varepsilon(u) = \left(\varepsilon_{ij}(u)\right)_{i,j=\overline{1,d}} \; ;$$
$$\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \; ;$$
$$div\, \sigma = (\sigma_{ij,j})_{i,j=\overline{1,d}} \; .$$

Also, we denote the real Hilbert spaces $\mathbb{H}$, $\mathcal{H}$, $\mathbb{H}_1$ and $\mathcal{H}_1$ respectively:

$$\begin{aligned} &H := L^2(\Omega) \; ; \; \mathbb{H} := L^2(\Omega)^d \; ; \\ &H_1 := H^1(\Omega) \; ; \\ &\mathcal{H} := \{\sigma = (\sigma_{ij}) \in \mathcal{S}_d : \\ &\qquad \sigma_{ij} \in L^2(\Omega)\} = \mathcal{S}_d(H) ; \\ &\mathcal{H}_1 := \{\sigma \in \mathcal{H} : \operatorname{div} \sigma \in \mathbb{H}\} ; \\ &\mathbb{H}_1 := \{u \in \mathbb{H} : \varepsilon(u) \in \mathcal{H}\} \end{aligned} \quad (2.1)$$

and the cannonical inner products, defined by:

$$(u,v)_H = \int_\Omega u \cdot v\, dx \; ;$$
$$(u,v)_{\mathbb{H}} = \int_\Omega u \cdot v\, dx \; ;$$
$$(u,v)_{H_1} = \int_\Omega u \cdot v\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx \; ;$$
$$\|u\|^2_{\mathbb{H}} = (u,u)_{\mathbb{H}} \; ; \quad \|u\|^2_{H_1} = (u,u)_{H_1} \; ;$$
$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma(\varepsilon(u)) : \tau(\varepsilon(u))\, dx = \int_\Omega \sigma_{ij}\tau_{ij}\, dx \; ;$$
$$\|\sigma\|^2_{\mathcal{H}} = (\sigma, \sigma)_{\mathcal{H}} \; ;$$
$$((u,v))_{\mathbb{H}_1} = (u,v)_{\mathbb{H}} + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \; ;$$
$$\|u\|^2_{\mathbb{H}_1} = ((u,u))_{\mathbb{H}_1} \; ;$$
$$((\sigma, \tau))_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\operatorname{div}\sigma, \operatorname{div}\tau)_{\mathbb{H}} \; ;$$
$$\|\sigma\|^2_{\mathcal{H}_1} = ((\sigma,\sigma))_{\mathcal{H}_1} \; .$$

We denoted by $\|\cdot\|_{\mathbb{H}}$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathbb{H}_1}$ and $\|\cdot\|_{\mathcal{H}_1}$ the cannonical norms induced by coresponding inner products in the respectively spaces.

For every element $v \in \mathbb{H}_1$ we also use the notation v to denote the trace of v or Γ (i.e. $\gamma_0 v = v|_\Gamma$) and, we denote by v_n and v_τ the *normal* and the *tangential components* of v on Γ given by,

$$v_n = v \cdot n \; ; \; v_\tau = v - v_n n \qquad (2.2)$$

We also denote by σ_n and σ_τ the *normal* and the *tangential traces* of the element $\sigma \in \mathcal{H}_1$ given by,

$$\sigma_n = (\sigma \cdot n) \cdot n \; ; \; \sigma_\tau = \sigma n - \sigma_n n \qquad (2.3)$$

We recall that the following *Green's formula* holds: for a regular function $\sigma \in \mathcal{H}_1$ fixed, and $(\forall)\, v \in \mathbb{H}_1$

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\operatorname{div}\sigma, v)_{\mathbb{H}} = - \langle \sigma n, v \rangle_{H^{-\frac{1}{2}}(\Gamma)^d,\, H^{\frac{1}{2}}(\Gamma)^d} \qquad (2.4)$$

We remember that the elastic-viscoplastic body is occupies of the regular domain $\Omega \subset \mathbb{R}^d$ with the surface Γ that is a *sufficiently regular* boundary, portionned into three disjoint measurable part,

$$\Gamma = \Gamma_u \cup \Gamma_\sigma \cup \Gamma_c$$

such that $ds - meas\,(\Gamma_u) > 0$.

Thus, we define the closed subspaces $\mathbb{V}$ and U of $\mathbb{H}_1$ and H_1, respectively, by:

$$\mathbb{V} := \{v \in \mathbb{H}_1 : v = 0, \; \text{on } \Gamma_u\} \qquad (2.5)$$

$$U := \{\theta \in H_1 : \theta = 0 \;, \text{on } \Gamma_u \cup \Gamma_\sigma\} \qquad (2.6)$$

and $\mathbb{K}$ be the convexe set of admisible displacements given by,

$$\mathbb{K} := \{v \in \mathbb{V} : v_n \leq 0 \;, \text{on } \Gamma_c\} \qquad (2.7)$$

Since $ds - meas\,(\Gamma_u) > 0$, Korn's inequality holds (see [9], 1997 - pp.291) and there exists $\alpha > 0$, depending only on Ω and Γ_u, such that:

$$((\sigma, \tau))_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\operatorname{div}\sigma, \operatorname{div}\tau)_{\mathbb{H}} \; ;$$
$$\|\sigma\|^2_{\mathcal{H}_1} = ((\sigma,\sigma))_{\mathcal{H}_1} \; .$$
$$\|v\|_{\mathbb{H}_1} \leq \alpha \|\varepsilon(v)\|_{\mathcal{H}} \; , \; (\forall)\, v \in \mathbb{V} \; . \qquad (2.8)$$

Hence, on $\mathbb{V}$ we consider the inner product given by,

$$(u,v)_{\mathbb{V}} = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \; , \; (\forall)\, u, v \in \mathbb{V} \qquad (2.9)$$

and the associated norm,

$$\|v\|_{\mathbb{V}} = \|\varepsilon(v)\|_{\mathcal{H}} \; , \; (\forall)\, v \in \mathbb{V} \; .$$

It follows that $\|\cdot\|_{\mathbb{H}_1}$ and $\|\cdot\|_{\mathbb{V}}$ are equivalent norms on $\mathbb{V}$ and therefore $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ is a real Hilbert space.

Moreover, by the Soblev's trace theorem and (2.9), we have constant $\gamma > 0$ depending only on the domain Ω and the its boundaries Γ_u and Γ_c such that,

$$\|v\|_{L^2(\Gamma_0)^d} \le \gamma \|v\|_{\mathbb{H}_1}, \quad (\forall)\, v \in \mathbb{V} \quad (2.10)$$

In an analogous way, we can prove that the norm $\|\theta\|_U \le \|\nabla\theta\|_H$, $(\forall)\, \theta \in U$ associated to the inner product on U given by $(\theta,\eta)_U = (\nabla\theta,\nabla\eta)_H$ is equivalent to the classical norm on H_1.

Hence $(U, \|\cdot\|_U)$ is a real Hilbert spaces. We also recall, that for every real Banach spaces E we use the notation $C^0([0,T];E)$ and $C^1([0,T];E)$ for the space of continuous and continuously differentiable function from $[0,T]$ to E, respectively, $C^0([0,T];E)$ is a real Banach space with the norm ,

$$\|u\|_{C^0([0,T];E)} = \max_{t\in[0,T]}\|u(t)\|_E \quad (2.11)$$

while $C^1([0,T];E)$ is the real Banach space with the norm,

$$\|u\|_{C^1([0,T];E)} = \|u\|_{C^0([0,T];E)} + \|\dot{u}\|_{C^0([0,T];E)} \quad (2.12)$$

If $k \in \mathbb{N}$ and $p \in [1,\infty]$ are arbitrary, then we use the standard notation for the Lebesgue spaces $L^p(0,T;E)$ and for the Sobolev spaces $W^{k,p}(0,T;E)$,

$$\|u\|_{W^{k,p}(0,T;E)} = \sum_{|m|\le k}\|D^m u\|_{L^p(0,T;E)} \quad (2.13)$$

While the Banach spaces E is $W^{k,p}(\Omega)$ we have,

$$\|u(t)\|_E = \|u(t)\|_{k,p,\Omega} = \sum_{|m|\le k}\|D^m u(t)\|_{L^p(\Omega)} = \sum_{|m|\le k}\left(\int_\Omega \|D^m u(t)(x)\|^p\,dx\right)^{\frac{1}{p}} \quad (2.14)$$
$$t \in [0,T];\ m \in \mathbb{N}^k$$

In this paper we use as an example $E = \mathbb{V}$, and we also use the Sobolev space $W^{1,\infty}(0,T;\mathbb{V})$ equipped with the norm ,

$$\|u\|_{W^{1,\infty}(0,T;\mathbb{V})} = \|u\|_{L^\infty(0,T;\mathbb{V})} + \|\dot{u}\|_{L^\infty(0,T;\mathbb{V})} = \operatorname{ess\,sup}_{t\in[0,T]}\|u(t)\|_{\mathbb{V}} + \operatorname{ess\,sup}_{t\in[0,T]}\|\dot{u}(t)\|_{\mathbb{V}} = \operatorname{ess\,sup}_{t\in[0,T]}\|\varepsilon(u(t))\|_{\mathcal{H}} + \operatorname{ess\,sup}_{t\in[0,T]}\|\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \quad (2.15)$$

Finally, we recall the following abstract result concerning some evolution equations (see [2], 1997- pp.151; [9], 1997- pp.124), and which will be used in *Section 3.* of this paper.

Theorem 2.1

Let $V \subset H \subset V'$ be a *Gelfand triple* , and $A: V \to V'$ is a *hemicontinuous* and *monotone* operator, i.e. $(\exists)\, \alpha,\gamma,\delta \in \mathbb{R}, (\alpha,\gamma > 0)$ such that,

$$< Av, v >_{V',V} \ge \alpha\|v\|_V^2 + \delta, \quad (\forall) v \in \mathbb{V} \quad (2.16)$$

$$\|Av\|_{V'} \le \gamma(\|v\|_V + 1), \quad (\forall)\, v \in \mathbb{V}. \quad (2.17)$$

If $u_0 \in H$ and $f \in L^2(0,T;V')$ are given functions, then the *evolution operatorial problem* ,

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), & t \in (0,T) \\ u(0) = u_0 \end{cases} \quad (2.18)$$

has a *unique solution* which satisfies the L^2-regularity properties,

$$u \in L^2(0,T;V) \cap C^0([0,T];H) \quad (2.19)$$
$$\dot{u} \in L^2(0,T;V').$$

We recall the Green formula, which is valid in this regular functional context :

$$(\sigma(\varepsilon(u)), \varepsilon(v))_{\mathcal{H}} + (div\,\sigma(\varepsilon(u)), v)_{\mathbb{H}} = < \sigma(\varepsilon(u))\cdot n, v >_{\mathbb{Y}',\mathbb{Y}}, \quad (\forall)\, v \in \mathbb{H}_1$$

where, we denote the functional spaces :

$X = L^2(\Gamma)$; $Y = H^{\frac{1}{2}}(\Gamma)$;
$X_i = L^2(\Gamma_i)$; $Y_i = H^{\frac{1}{2}}(\Gamma_i)$; $(i = u, \sigma, c)$
$\mathbb{X} = L^2(\Gamma)^d$; $\mathbb{Y} = H^{\frac{1}{2}}(\Gamma)^d$;
$\mathbb{X}_i = L^2(\Gamma_i)^d$; $\mathbb{Y}_i = H^{\frac{1}{2}}(\Gamma_i)^d$; $(i = u, \sigma, c)$
$Y' = H^{-\frac{1}{2}}(\Gamma)$; $\mathbb{Y}' = H^{-\frac{1}{2}}(\Gamma)^d$;
$Y_i' = H^{-\frac{1}{2}}(\Gamma_i)$; $\mathbb{Y}_i' = H^{-\frac{1}{2}}(\Gamma_i)^d$; $(i = u, \sigma, c)$

and obtained,

$$\sigma(u)\cdot n \in \mathbb{Y}', \ (\forall)\, \sigma \in \mathcal{H}_1.$$

Let $\mathbb{V}$ denote the closed subspace of $\mathbb{H}_1$ defined by,

$$\mathbb{V} = \{ v \in \mathbb{H}_1 : v = 0, on\ \Gamma_u \} \quad (2.20)$$

We note that the *Korn inequality* holds (see [9], 1997-pp. 96), i.e. exist $\alpha > 0$ only depend by Ω *and* Γ_u :

$$\|\varepsilon(u)\|_{\mathcal{H}} \ge \alpha\|u\|_{\mathbb{H}_1}, \quad (\forall)\, u \in \mathbb{V}.$$

By using the inner product on $\mathbb{V}$,

$$(u, v)_{\mathbb{V}} = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

and the norm induced $\|u\|_{\mathbb{V}} = (u, u)_{\mathbb{V}}$ we obtain that $\mathbb{V}$ is a Hilbert space.

Next, we denote by $t \to F(t)$ an element of $\mathbb{V}'$ given by:

$$< F(t), v >_{\mathbb{V}', \mathbb{V}} = (f(t), v)_{\mathbb{H}} + (g(t), \gamma_0 v)_{\mathbb{X}_\sigma}, \ (\forall)\, v \in \mathbb{V},\ t \in [0,T] \tag{2.21}$$

where, f and g are input data, and $\gamma_0 v \in \mathbb{Y}_\sigma \subset \mathbb{X}_\sigma$ is the trace over Γ of the vector $v \in \mathbb{V} \subset \mathbb{H}_1$.

For the *variational* description of the *friction law*, we denote the *friction functional*,

$$j : \mathcal{H}_1 \times \mathbb{V} \to \mathbb{R}\ ,$$
$$j(\sigma, v) = \int_{\Gamma_c} |\mu|\, |\mathcal{T}\sigma_n|\, |v_\tau|\, ds \tag{2.22}$$

where,

$|\mu| = \|\mu\|_{\mathbb{X}_c}$; $|v_\tau| = \|\gamma_0 v - \gamma_2 v\|_{\mathbb{X}_c}$;

$\mathcal{T} : \mathbb{Y}_c' = H^{-\frac{1}{2}}(\Gamma_c)^d \to \mathbb{X}_c = L^2(\Gamma_c)^d$ i.e.

$|\mathcal{T}\sigma_n| = \|\mathcal{T}\sigma_n(v)\|_{\mathbb{X}_c}$;

Thus, for the *variational* description of the *unilateral contact condition*, we introduce the space $\mathbb{Y}_c = H^{\frac{1}{2}}(\Gamma_c)$ as the set of restrictions to Γ_c of the $\mathbb{Y} = H^{\frac{1}{2}}(\Gamma)$ functions which are null on Γ_u.

Also, we denote by $< \cdot , \cdot >_{\mathbb{V}', \mathbb{V}}$ the duality pairing between $\mathbb{Y}_c = H^{\frac{1}{2}}(\Gamma_c)$ and its dual $\mathbb{Y}_c{}' = H^{-\frac{1}{2}}(\Gamma_c)$,

$$< \sigma_n(t), v_n >_{\mathbb{Y}_c', \mathbb{Y}_c} = \int_{\Gamma_c} \sigma_n(t)\, v_n\, ds, \tag{2.23}$$
$$(\forall)\, v \in \mathbb{V},\ \sigma \in \mathcal{H}_1,\ \text{a.p.t.}\ t \in (0,T).$$

Now, let us introduce the *convex* set of admisible displacements, defined by

$$\mathbb{K} = \{ v \in \mathbb{V} : v_n \le 0\ , \text{on}\ \Gamma_c \}. \tag{2.24}$$

Finally, in the study of the thermo-mechanical problem $(P) \equiv (1.6) - (1.17)$ we assume that the operators $\mathcal{A}$, $\mathcal{G}$ satisfies some regularity conditons ([5], [6], [12]).

Thus, the *variational formulation* for thermo-mechanical problem $(P) \equiv (1.6) - (1.17)$ is obtained.

Problem (VP) : Find,

a *displacement field* $u : [0,T] \to \mathbb{V}$,

a *stress field* $\sigma : [0,T] \to \mathcal{H}$ which satisfy the *evolutionary quasivariational problem*:

$$\sigma = \mathcal{A}(\varepsilon(\dot{u})) + \mathcal{G}(\varepsilon(u)),\ t \in (0,T) \tag{2.25}$$

$$\begin{aligned} &(\sigma(t)\ ,\ \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathbb{H}} + \\ &+ j(\sigma(t), v) - j(\sigma(t), \dot{u}(t)) \ge \\ &\ge < F(t), v - \dot{u}(t) >_{\mathbb{V}', \mathbb{V}} + \\ &< \sigma_n(t), v_n - \dot{u}_n(t) >_{\mathbb{Y}_c', \mathbb{Y}_c}, \\ &(\forall)\, v \in \mathbb{V},\ \text{a.p.t.} \end{aligned} \tag{2.26}$$

$t \in (0,T)$.

$$< \sigma_n(t), w_n - u_n(t) >_{\mathbb{Y}_c', \mathbb{Y}_c} \quad (\forall)\, w \in \mathbb{K},\ (\forall)\, t \in [0,T] \tag{2.27}$$

$$u(0) = u_0 \tag{2.28}$$

We assume that the *input data* of *quasivariational problem* (**VP**) ≡ (2.25)-
–(2.28) satisfies the minimum regularity conditions,

$$f \in L^\infty(0,T;\mathbb{H})\ ;\ g \in L^\infty(0,T;\mathbb{X}_\sigma)\quad u_0 \in \mathbb{K} \tag{2.29}$$

Theorem 2.2 ([7])

(for the *existence* of *weak solution* $\{ u : \sigma \}$)

Assume that the input data satisfies the minimum regularity conditions (2.29) , and the regularity hypotesis for the *elasticity operator* $\mathcal{G}$ and for the *viscoplasticity operator* $\mathcal{A}$ respectively, to hold good.

Then, there exist a *weak solution* $\{ u : \sigma \}$ to the problem (**VP**) ≡(2.25)–(2.28) satisfying the minimum regularity conditions,

$$u \in W^{1,\infty}(0,T;\mathbb{V}) \cap C([0,T];\mathbb{K}),\quad \sigma \in L^\infty(0,T;\mathcal{H}_1). \tag{2.30}$$

Remark 2.3

We could have taken data f and g with the L^2 – *regularity properties* in the time variable,

$$f \in L^2(0,T;\mathbb{H})\ ;\ g \in L^2(0,T;\mathbb{X}_\sigma) \tag{2.31}$$

and $u_0 \in \mathbb{K}$, then we would obtain the existence of weak solutions for the problem (**VP**), also, satisfying the L^2–*regularity properties*,

$$u \in H^1(0,T;\mathbb{K})\ ;\ \sigma \in L^2(0,T;\mathcal{H}_1) \tag{2.32}$$

3. Main existence and uniqueness results

In this section we use the *temporal semi-discretization Galerkin method* for the proof of the existence concerning the weak solutions of the (**VP**) problem, (see [5], [6], [12]). For this, we need the following notations, which assume to introduce a new variational formulation of the initial problem (**P**).

We define the following functionals,

$a: \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ a *bilinear form* by,

$$a(u,v) = \left(\mathcal{A}(\varepsilon(u)),\ \varepsilon(v)\right)_{\mathcal{H}}; \qquad (3.1)$$

$b: \mathbb{V} \times \mathbb{V} \to \mathbb{R}$, a *linear form* only with respect to the second argument given by,

$$b(u,v) = \left(\mathcal{G}(\varepsilon(u)),\ \varepsilon(v)\right)_{\mathcal{H}}; \qquad (3.2)$$

$l: \mathbb{V} \to \mathbb{R}$ *linear form*,

$$l(v) = (f(t), v)_{\mathbb{H}} + (g(t), \gamma_0 v)_{\mathbb{H}_\sigma} + < \sigma_n(t), v_n >_{Y_c', Y_c} = \int_\Omega f(t) v dx + \int_{\Gamma_\sigma} g(t) v d\sigma + \int_{\Gamma_c} \sigma_n(t) v_n d\sigma \qquad (3.3)$$

a.p.t. $t \in (0,T)$;

Thus, the *quasi-variational evolutionary problem* (**VP**) ≡(2.25)−(2.28) find,

a *displacement field* $u: [0,T] \to \mathbb{V}$, and

a *stress field* $\sigma: [0,T] \to \mathcal{H}$

which satisfies the following *problem for a quasi-variational inequality*,

$$\sigma = \mathcal{A}(\varepsilon(\dot{u}) + \mathcal{G}(\varepsilon(u))),\ t \in (0,T) \qquad (3.4)$$

$$a(\dot{u}(t),\ v - \dot{u}(t)) + b(u(t),\ v - \dot{u}(t)) + j(\sigma(t),\ v) - j(\sigma(t),\ \dot{u}(t)) \geq l(v - \dot{u}(t)), \quad (\forall)\, v \in \mathbb{V},\ \text{a.p.t. } t \in (0,T) \qquad (3.5)$$

$$< \sigma_n(t), v_n - u_n(t) >_{Y_c', Y_c} \geq 0, \quad (\forall)\, v \in \mathbb{K},\ (\forall)\, t \in [0,T] \qquad (3.6)$$

$$u(0) = u_0. \qquad (3.7)$$

4. Conclusions

In the present paper has been investigated a mathematical model for as well triboprocess involving the coupling thermal and mechanical aspects by specific behaviour laws of materials.

The contact condition for this quasistatic processes has described as effect of a normal and tangential damped response properties.

The *classical* as well as a *variational formulations* of the thermo-mechanical problem are presented.

References

[1]. Amassad A., Kuttler L., Rochdi M., Shillor M. - *Quasistatic thermo-viscoelastic contact problem with slip dependent friction coefficient*, Math. Computational Modelling, No. **36**, pp.839-854, 2007

[2]. Andrews K.T., Kuttler K.L, Shillor M. - *On the Dynamic behaviour of a Thermo-Viscoplastic Body in Frictional Contact with a Rigid Obstacle*, European J. Applied Mathematics , vol.**8**, pp.417-436, 1997

[3]. Ayyad Y., Sofonea M. - *Analysis of two Dynamic Frictionless Contact Problems for Elastic-Visco-Plastic Materials*, EJDE , No.**55** , pp.1-17, 2007

[4]. Barbu V. - *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Bucharest–Noordhoff, Leyden, 1976

[5]. Bendrea C. - *Variational evolutionary problems and quasi-variational inequalities for mathematical modeling of the tribological processes concerning continuous casting machine*, Thesis, Galati, 2008

[6]. Bendrea C., **Munteanu V**. - *Thermal Analysis of an Elastic-Viscoplastic Unilateral Contact Problem in the Continuous Casting of the Steel*, Metalurgia International, Vol XIV (2009), No **5**, pp. 72

[7]. Campo M., Fernandez J.R. - *Numerical analysis of a quasistatic thermo—viscoelastic frictional contact problem*, Comput.Math., No.**35**, pp.453- 469, 2005

[8]. Chau O., **Fernandez J.R**., **Han W**., **Sofonea M**. - *A frictionless contact problem for elastic-viscoplastic materials with normal compliance and damage*, Comput. Methods Appl. Mech. Engrg., **191**, pp.5007-5026, 2002

[9]. Ciarlet P.G. - *Mathematical Elasticity* (*vol.* I, II), Elsevier , Amsterdam – New-York, 1997

[10]. Fernandez J.R, Sofonea M., Viano J.M. - *A Frictionless Contact Problem for Elastic-Viscoplastic Matherials with Normal Compliance: Numerical Analysis and Computational Experiments*, Comput. Methods Appl. Mech. Engrg., vol. **90**, pp.689-719, 2002

[11]. Han W., Sofonea M. - *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticiy*, AMS , Int. Press, 2001

[12]. Moitra A.,Thomas B.G. - *Applications of a Thermo-Mechanical Finite Element Method of Steel Shell Behaviour in the Continuous Slab Casting Mold*, SteelMaking Proceedings, vol.**76**, pp.657-667, 1998

[13]. Munteanu V., **Bendrea C**. - *A Thermoelastic Unilateral Contact Problem with Damage and Wear in Solidification Processes Modeling*, Metalurgia International, Vol XIV (2009), No **5**, pp. 83

[14]. Thomas B.G., O'Malley R., Stone G. - *Measurement of Temperature , Solidification and Microstructure in a Continuous Cast Thin Slab* , ISIJ Int., vol.**46**, No.**10** , pp.1394-1415, 2006

[15]. Wang H., Li G.; et. al.- *Mathematical Heat Transfer Model Research for the Improvement of Continuous Casting Slab Temperature*, ISIJ Int. vol.**45**, No.**9**, pp.1291-1296.2005