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A THERMODYNAMIC CONTACT PROBLEM FOR ELASTIC-VISCOPLASTIC MATERIALS

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ABSTRACT

Mathematical modeling of stress generation and heat transfer in casting processes is a difficult and complex subject that is now receiving increased attention. In this paper, we study a dynamic problem which describes the frictional contact between an elastic-viscoplastic body (*a deformable product of casting process*: *slab, bloom etc.*) *and a rigid obstacle* (*walls of the mold or cilindrical rolls system employed for support – traction – soft reduction*) *in complicated conditions concerning heat conduction on contact interface in the solidification process.*

In the next sections we will briefly formulate the fundamentals of the kinematics for a large deformation approach and the basic equations governing our model. After this, we will explain, in more details, the classical problem statement and the foundation of the weak formulation of this problem, which consist of certain variational inequalities for the viscoelasticity and viscoplasticity parts, and, a variational parabolic equation for the heat conduction part.

KEYWORDS: elastic-viscoplastic materials, thermal conduction, dynamic contact, variational inequality, continuous casting

1. Introduction

In the continuous casting process, illustrated by a schematic representation in *Figure1* molten metal is poured from the ladle into the tundish and then through a submerged entry nozzle into a mould cavity.

The mould is water-cooled so that enough heat is extracted to solidify a shell of sufficient thickness. The shell is withdrawn from the bottom of the mould at a "casting speed" that matches the inflow of metal, so that the process ideally operates at steady state. Below the mould, water is sprayed to further extract heat from the strand surface, and the strand eventually becomes fully solid when it reaches the "metallurgical length".

Solidification begins in the mould, and continues through the different zones of cooling while the strand is continuously withdrawn at the casting speed. Finally, the solidified strand is straightened, cut and then discharged for intermediate storage or hot charged for finished rolling.

The paper is organized as follows:

- In *Section 2* we present the statement of the thermo-mechanical problem and its variational formulation;
- In *Section 3* we propose our main existence and uniqueness results;
- *Section 4* is reserved to the concluding remarks;

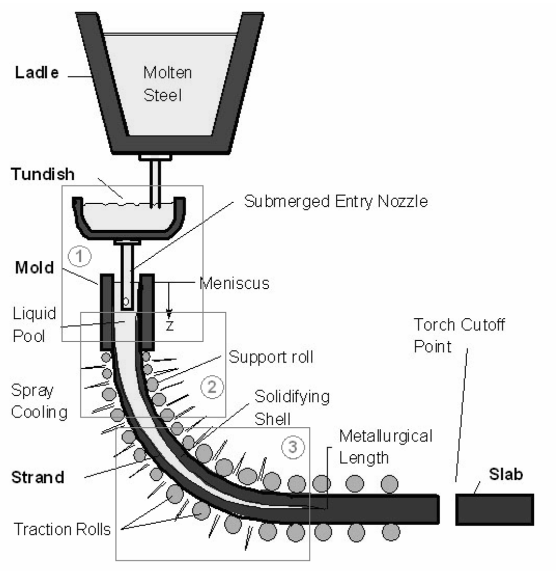

Fig. 1. *Schematic representation of Continuous Casting Process [6], [12].*

1. Schematic of tundish and mould region of continuous casting process and of specific contacts;
2. Thermo-elastic-viscoplastic contact between slab and the support- rolls on secondary cooling zones (detail in *Figure 2.*);
3. The contact between slab and the traction-soft reduction rolls on secondary cooling zones;

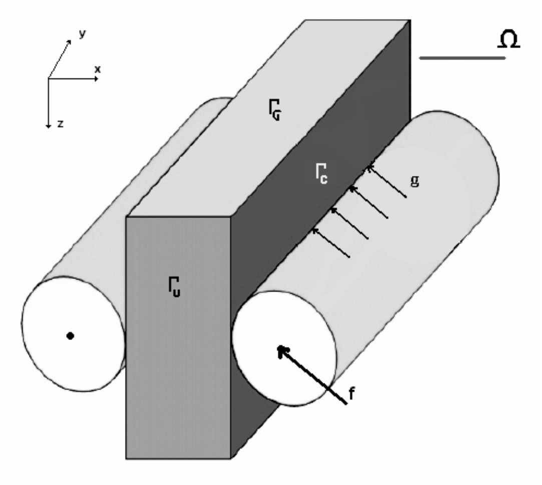

***Fig. 2.** The contact between slab and support rolls.*

2. Problem statement and variational formulation

In earlier mathematical publications there were several simplifications assumed recording to which the deformable bodies were linearly elastic. However, numerous recent studies are dedicated to the modeling, variational analysis and numerical approximations of contact problems involving rheological properties of the materials.

Moreover, a variety of new and modified contact conditions have been employed, reflecting a variety of possible physical contact settings and conditions.

Now, we describe the model for the physics process and derive its *weak* (*variational*) formulation.

An elastic-viscoplastic body (slab, blum, etc.) occupies a regular domain $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ with surface Γ that is portioned into three disjoint measurable parts $\Gamma = \Gamma_u \cup \Gamma_\sigma \cup \Gamma_c$ such that means (Γ_u) > 0. Let $[0,T]$ be the time interval of interest with $T > 0$. The body is clamped on $(0,T) \times \Gamma_u$ and therefore the displacement field vanishes there. We denote by S_d the spaces of second order symmetric tensors, while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d or $\mathbb{R}^d$. Let n denote the unit outer normal on Γ, and everywhere in the sequel the index i,j runs from 1 to d (summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable).

We also use the following notation and physical nomenclatures:

$\Omega_T = (0,T) x\, \Omega$;

$\overline{\Omega}_T = [0,T] \times (\Omega \cup \Gamma)$

$\Gamma = \partial\Omega$; $\Gamma_T = (0,T) x\, \Gamma$;

$\Gamma_{iT} = (0,T) x\, \Gamma_i$ for $i \in \{u\ ;\ \sigma\ ;\ c\}$;

$t \in [0,T]$ time variable;

$x \in \Omega$ spatial variable;

u : $\overline{\Omega}_T$ → ℝd displacement vectorial field;

$\dot{u} = \left(\frac{\partial u_i}{\partial t}\right)$; $\ddot{u} = \left(\frac{\partial^2 u_i}{\partial t^2}\right)$ velocity and inertial vectorial fields;

$\sigma : \overline{\Omega}_T \to S_d$ *stress tensor field* (second order Piola –Kirchhoff) ;

$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ *strain tensor field* (linearized tensor Green-St. Venant);

$\theta : \overline{\Omega}_T \to \mathbb{R}$ *temperature scalar field*;

The aim of this paper is to study a *thermodynamic contact problem* for *elastic-viscoplastic materials* with a constitutive law of the form (2.1), where $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{B}$ are *nonlinear* operators whitch will be described below, and $C = (c_{ij})$ represents the *thermal expansion tensor*.

Here and below, in order to simplify the notation, we usually do not indicate explicitly the dependence of the functions on the variables $x \in \overline{\Omega}$ (on the time $t \in [0,T]$ sometimes). Examples of constitutive laws of the form (2.1) can be constructed by using *thermal* aspects and *rheological* arguments, see e.g. [10], [8], [14], [7].

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{G}\varepsilon(u(t)) - C\theta(t) + \int_0^t \mathcal{B}\Big(\sigma(s) - \mathcal{A}\varepsilon(\dot{u}(s)) + C\theta(s)\ .\ \varepsilon(u(s))\Big)\, ds \qquad (2.1)$$

It follows from (2.1) that, at each time moment t, the stress tensor $\sigma(t)$ is split into two parts,

$$\sigma(t) = \sigma^v(t) + \sigma^{evp}(t) \text{ where,}$$

$$\sigma^v(t) = \mathcal{A}\varepsilon(\dot{u}(t)), \qquad (2.2)$$

is the *purely viscous* part, and

$$\sigma^{evp}(t) = \mathcal{G}\varepsilon(u(t)) - \mathcal{C}\theta(t) + \int_0^t \mathcal{B}\left(\sigma^{evp}(s) - \mathcal{A}\varepsilon(\dot{u}(s)), \varepsilon(u(s))\right), \text{ in } \Omega_T \quad (2.3)$$

is the *rate-type elastic-viscoplastic* part.

When $\mathcal{B} = 0$, $\mathcal{C} = O_d$ the constitutive law (2.1) reduces to the *Kelvin-Voigt viscoelastic* behaviour of the materials,

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{G}\varepsilon(u) \quad , \quad \text{in } \Omega_T \quad (2.4)$$

We turn to describe the frictional contact conditions. Since the linear unilateral contact at high temperature between deformable body $\overline{\Omega}$ (product of the cast) and the rigid obstacle (walls of the mold, or traction-support rolls, respectively) is lubricated all trough of the solidification process, we assume that the normal stress $\sigma_n = n^T \sigma n$ on the contact surface Γ_c can be satisfies through the following semilinear relation (see [5], [7]) ,

$$-\sigma_n = p_n(\dot{u}_n) \quad , \quad \text{on } \Gamma_{c;T} \quad (2.5)$$

The *normal damped response* function p_n is prescribed and satisfies $p_n(\dot{u}_n) = 0$ for $\dot{u}_n \le 0$, since then there is no contact. As an example, we may consider (see [10], [7])

$$p_n(r) = \delta_n r_+ \quad , \quad (\forall)\, r \in \mathbb{R} \quad (2.6)$$

where $\delta_n > 0$ represents a deformability positive coefficient and $r_+ = max\{0\,;r\}$.

Also, we consider the following associated *friction law*,

$$-\sigma_\tau = p_\tau(\dot{u}_\tau), \quad \text{on } \Gamma_{c;T} \; . \quad (2.7)$$

As an example, we may consider the following form of *tangential damped response* function p_τ,

$$p_\tau(r) = \mu_\tau r, \quad (\forall)\, r \in \mathbb{R}^d \quad (2.8)$$

where μ_τ is a *frictional coefficient* and the *tangential shear stresses* σ_τ is proportional to the *tangential velocity* $\dot{u}_\tau$ (the setting where the contact surface is lubricated with a thin layer of a *newtonian fluid*).

Both p_τ and p_n are *friction-contact constitutive functions* whose properties will be described below.

Finally, the evolution of the temperature field is governed by the *heat transfer equation* (see [6], [8], [9]),

$$\dot{\theta} - div(K\nabla\theta) = q - \mathcal{C}\cdot\nabla\dot{u}, \text{ in } \Omega_T \quad (2.9)$$

where,

$K = (k_{ij})_{i,j=\overline{1,d}}$ is *thermal conductivity* tensor

$\mathcal{C} = (c_{ij})_{i,j=\overline{1,d}}$ is *thermal expansion* tensor;

q represent the *density of volume heat sources*.

In order to simplify the description of the problem, a *homogeneous condition* for the temperature field is considered on $\Gamma_u \cup \Gamma_\sigma$,

$$\theta = 0 \quad , \quad \text{on } \Gamma_{u;T} \cup \Gamma_{\sigma;T} \quad (2.10)$$

It is straightforward to extend the results shown in this paper to more general cases.

Also, we assume the associated *temperature boundary condition* is described on Γ_c ,

$$(K\nabla\theta)\cdot n = -k_e(\theta - \theta_r), \text{ on } \Gamma_{c;T} \quad (2.11)$$

where θ_r is the reference temperature of the obstacle, and k_e is the heat excange coefficient between the body and the rigid foundation.

Thus, the *classic thermo-mechanical problem* corresponding to the *thermo-cvasistatic contact* of an *elastic-viscoplastic* body with a rigid foundation, involving the *friction* and the *heat conduction*, can be written as follows:

Problem (P): Find,

a displacement field $u : \overline{\Omega}_T \to \mathbb{R}^d$,

a stress tensor field $\sigma : \overline{\Omega}_T \to \mathbb{S}_d$ and,

a temperature field $\theta : \overline{\Omega}_T \to \mathbb{R}$ such that,

$$\sigma(t) := \mathfrak{F}\left(\varepsilon(u(t)), \theta(t)\right) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{G}\varepsilon(u(t)) - \mathcal{C}\theta(t) + \int_0^t \mathcal{B}\left(\sigma(s) - \mathcal{A}\varepsilon(\dot{u}(s)) + \right. \quad (2.12)$$

$$\rho\ddot{u} - div\,\sigma = f \; , \quad \text{in } \Omega_T \quad (2.13)$$

$$\dot{\theta} - div\,(K\nabla\theta) = q - \mathcal{C}\cdot\nabla\dot{u}, \text{ in } \Omega_T \quad (2.14)$$

$$u = 0 \; , \quad \text{on } \Gamma_{u;T} = (0,T)\times\Gamma_u \quad (2.15)$$

$$\sigma_n = g \; , \quad \text{on } \Gamma_{\sigma;T} = (0,T)\times\Gamma_\sigma \quad (2.16)$$

$$\theta = \theta_r \; , \quad \text{on } \Gamma_{u;T} \cup \Gamma_{\sigma;T} \quad (2.17)$$

$$-\sigma_n = p_n(\dot{u}_n)\,;\; -\sigma_\tau = p_\tau(\dot{u}_\tau), \text{ on } \Gamma_{c;T} = (0,T)\times\Gamma_c \quad (2.18)$$

$$(K\nabla\theta)\cdot n = -k_e(\theta - \theta_r), \text{ on } \Gamma_{c;T} \quad (2.19)$$

$$u(0) = u_0\,;\; \dot{u}(0) = v_0\,;\; \theta(0) = \theta_0 \quad (2.20)$$

Here, u_0 and θ_0 represent the initial displacement and the initial temperature, respectively. Also, v_0 is the initial velocity of displacement. A volume force of density f acts in Ω_T and a surface traction of density g acts on $\Gamma_{\sigma;T}$.

In order to obtain the *variational formulation* of *Problem* ***(P)***, let us introduce additional notation and assumptions on the problem data, $\mathrm{H} := \mathrm{L}^2(\Omega)$;
$\mathbb{H} := \mathrm{L}^2(\Omega)^d$; $\mathrm{H}_1 := \mathrm{H}^1(\Omega)$; #
$\mathcal{H} := \{\boldsymbol{\sigma} = (\sigma_{ij}) \in \mathcal{S}_d \,:\, \sigma_{ij} \in \mathrm{L}^2(\Omega)\} = \mathcal{S}_d(H)$;
$\mathcal{H}_1 := \{\boldsymbol{\sigma} \in \mathcal{H} : \operatorname{div} \boldsymbol{\sigma} \in \mathbb{H}\}$;
$\mathbb{H}_1 := \{\boldsymbol{u} \in \mathbb{H} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\}$.

Here, $\boldsymbol{\varepsilon} : \mathbb{H}_1 \to \mathcal{H}_1$ and $div : \mathcal{H}_1 \to \mathbb{H}$ are the Hooke deformation and divergente operators, respectively.

The real Hilbert spaces $\mathbb{H}$,$\mathcal{H}$, $\mathbb{H}_1$ and $\mathcal{H}_1$ are endowed with the corresponding cannonical inner products,

$$(\boldsymbol{u},\boldsymbol{v})_{\mathbb{H}} := \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx = \int_\Omega u_i\cdot v_i\,dx$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} := \int_\Omega \boldsymbol{\sigma}:\boldsymbol{\tau}\,dx = \int_\Omega \sigma_{ij}\tau_{ij}\,dx$$
$$((\boldsymbol{u},\boldsymbol{v}))_{\mathbb{H}_1} := (\boldsymbol{u},\boldsymbol{v})_{\mathbb{H}} + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$
$$((\boldsymbol{\sigma},\boldsymbol{\tau}))_{\mathcal{H}_1} := (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{div}\boldsymbol{\sigma},\operatorname{div}\boldsymbol{\tau})_{\mathbb{H}}$$

We recall that the following *Green's formula* holds:

for a regular function $\boldsymbol{\sigma} \in \mathcal{H}_1$ fixed,

$$(\boldsymbol{\sigma},\,\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{div}\boldsymbol{\sigma},\,\boldsymbol{v})_{\mathbb{H}} = \\ = \langle \boldsymbol{\sigma n}\,,\,\boldsymbol{v} \rangle_{H^{-\frac{1}{2}}(\Gamma)^d\,,\;H^{\frac{1}{2}}(\Gamma)^d}, \quad (\forall)\,\boldsymbol{v} \in \mathbb{H}_1 \tag{2.21}$$

We remember that the elastic-viscoplastic body is occupied by the regular domain $\Omega \subset \mathbb{R}^d$ with the surface Γ that is a *sufficiently regular* boundary, portionned into three disjoint measurable part, $\Gamma = \Gamma_u \cup \Gamma_\sigma \cup \Gamma_c$ such that $\operatorname{meas}(\Gamma_u) > 0$.

Thus, we define the closed subspaces $\mathbb{V}$ and U of $\mathbb{H}_1$ and H_1, respectively, by:

$$\mathbb{V} := \{\boldsymbol{v} \in \mathbb{H}_1 : \boldsymbol{v} = \boldsymbol{0}, \text{on } \Gamma_u\}, \\ U := \{\theta \in \mathrm{H}_1 \,:\, \theta = 0 \text{ ,on } \Gamma_u \cup \Gamma_\sigma\} \tag{2.22}$$

and $\mathbb{K}$ be the convexe set of admisible displacements given by,

$$\mathbb{K} := \{\boldsymbol{v} \in \mathbb{V} \,:\, v_n \le 0 \,,\, \text{ on } \Gamma_c\}. \tag{2.23}$$

Since $\operatorname{meas}(\Gamma_u) > 0$, Korn's inequality holds (see [9], 1997-pp.291) and, hence, on $\mathbb{V}$ we consider the inner product given by:

$$(\boldsymbol{u},\boldsymbol{v})_{\mathbb{V}} = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \;\; (\forall)\,\boldsymbol{u},\boldsymbol{v} \in \mathbb{V} \tag{2.24}$$

and the associated norm,

$$\|\boldsymbol{v}\|_{\mathbb{V}} = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}},\;\; (\forall)\,\boldsymbol{v} \in \mathbb{V}$$

It follows that $\|\cdot\|_{\mathbb{H}_1}$ and $\|\cdot\|_{\mathbb{V}}$ are equivalent norms on $\mathbb{V}$ and therefore $(\mathbb{V}\,,\|\cdot\|_{\mathbb{V}})$ is a real Hilbert space.

In an analogous way, we can prove that the norm $\|\theta\|_U \le \|\nabla\theta\|_{\mathbb{H}}$, $(\forall)\,\theta \in U$ associated to the inner product on U given by $(\theta,\eta)_U = (\nabla\theta,\nabla\eta)_{\mathbb{H}}$ is equivalent to the classical norm on H_1. Hence $(U,\|\cdot\|_U)$ is a real Hilbert spaces.

We also recall (see [18], [4]), that for every real Banach spaces E we use the notation $C^0([0,T];E)$ and $C^1([0,T];E)$ for the space of continuous and continuously differentiable function from $[0,T]$ to E, respectively.

$C^0([0,T];E)$ and $C^1([0,T];E)$ are real Banach spaces with the norms,

$$\|\boldsymbol{u}\|_{C^0([0,T];E)} = \max_{t\in[0,T]}\|\boldsymbol{u}(t)\|_E\,, \\ \|\boldsymbol{u}\|_{C^1([0,T];E)} = \\ = \|\boldsymbol{u}\|_{C^0([0,T];E)} + \|\dot{\boldsymbol{u}}\|_{C^0([0,T];E)} \tag{2.25}$$

If $k \in \mathbb{N}$ and $p \in [1,\infty]$ are arbitrary, then we use the standard notation for the Lebesgue spaces $L^p(0,T\,;E)$ and for the Sobolev spaces $W^{k,p}(0,T\,;E)$. While the Banach spaces E is $W^{k,p}(\Omega)$ we have,

$$\|\boldsymbol{u}(t)\|_E = \|\boldsymbol{u}(t)\|_{k,p,\Omega} = \\ = \sum_{|m|\le k}\|D^m\boldsymbol{u}(t)\|_{L^p(\Omega)}, \\ t \in [0,T]\,;\; m \in \mathbb{N}^k. \tag{2.26}$$

In the study of the thermo-mechanical problem $(\boldsymbol{P}) \equiv (2.12) - (2.20)$ we assume that the *viscosity* operator $\mathcal{A}$, the *elasticity* operator $\mathcal{G}$, the *viscoplasticity* operator $\mathcal{B}$ and the *contact-friction functions* p_n , p_τ satisfies some regularity conditons (see [7], [2], [3]).

We assume the following $L^2 - \text{regularity}$ for the given force densities,

$$\boldsymbol{f} \in L^2(0,T\,;\mathbb{H})\,;\;\; \boldsymbol{g} \in L^2\big(0,T\,;L^2(\Gamma_\sigma)^d\big) \tag{2.27}$$

and that the *thermal conductivity* and *expansion tensors* are symmetrically bounded tensors satisfying,

$$c_{ij} = c_{ji} \in L^\infty(\Omega)\;;\; k_{ij} = k_{ji} \in L^\infty(\Omega) \\ k_{ij}\,\xi_i\xi_j \ge \alpha_K\xi_i^2\;;\; \xi \in \mathbb{R}^d, \alpha_K > 0. \tag{2.28}$$

Finally, we also suppose that the mass density satisfies,

$$\rho \in L^2(\Omega)\text{: } \rho(\boldsymbol{x}) \ge \rho_0, \quad a.e.\; \boldsymbol{x} \in \Omega \tag{2.29}$$

and the initial data satisfy,

$$\boldsymbol{u}_0 \in \mathbb{V}\,;\;\; \boldsymbol{v}_0 \in \mathbb{H}\,; \\ \theta_0 \in H\,;\;\; \theta_r \in L^2\big(0,T\,;L^2(\Gamma_c)\big). \tag{2.30}$$

Because the inclusion mapping of $(\mathbb{V}, |\cdot|_{\mathbb{V}})$ into $(\mathbb{H}, |\cdot|_{\mathbb{H}})$ and identifying $\mathbb{H}$ with its own dual we can write the Gelfand triple $\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}'$.

Let $F : [0,T] \to \mathbb{V}'$, $F \in L^2(0,T;\mathbb{V}')$,

$$< F(t), v >_{\mathbb{V}',\mathbb{V}} := (f(t), v)_{\mathbb{H}} + (g(t), v)_{L^2(\Gamma_\sigma)^d} = \\ = \int_\Omega f(t) \cdot v \, dx + \int_{\Gamma_\sigma} g(t) \cdot v \, ds, \quad (\forall)\, v \in \mathbb{V}, (\forall)\, t \in [0,T]. \tag{2.31}$$

We also consider the *contact-functionals*,

$$j : \mathbb{V} \times \mathbb{V} \to \mathbb{R}, \\ j(u(t), v) := \left(p_n(\dot{u}_n(t)), v_n\right)_{L^2(\Gamma_c)} + \\ + \left(p_\tau(\dot{u}_\tau(t)), v_\tau\right)_{L^2(\Gamma_c)} = \\ \int_{\Gamma_c} p_n(\dot{u}_n(t))\, v_n \, ds + \int_{\Gamma_c} p_\tau(\dot{u}_\tau(t)) \cdot v_\tau \, ds \quad (\forall)\, t \in [0,T]. \tag{2.32}$$

and $l : U \times U \to \mathbb{R}$ defined by,

$$l(\theta(t), \eta) := \\ = -\int_{\Gamma_c} k_c\left(\theta(t) - \theta_r(t)\right) \eta \, ds, \quad (\forall)\, t \in [0,T]. \tag{2.33}$$

We suppose in what follows that $\{u\, ;\, \sigma\, ;\, \theta\}$ are smooth functions satisfying the problem $(P) \equiv (2.12) - (2.20)$.

We take the dot product of equation (2.12) with $w \in \mathbb{V}$, for w an arbitrary element of $\mathbb{V}$, the integrate the result over Ω, and using Green’s formula (2.21) we obtain,

$$(\rho \ddot{u}(t), w)_{\mathbb{H}} + (\sigma(t), \varepsilon(w))_{\mathcal{H}} = \\ = (f(t), w)_{\mathbb{H}} + (\sigma(t) n, w)_{L^2(\Gamma)^d} \quad (\forall) w \in \mathbb{V}, \text{ a.e } \in (0,T). \tag{2.34}$$

Thus, the *variational formulation* for thermo-mechanical problem **(P)** is obtained.

Problem (VP): Find,

a *displacement field* $u : [0,T] \to \mathbb{V}$,

a *stress field* $\sigma : [0,T] \to \mathcal{H}$ and,

a *temperature field* $\theta : [0,T] \to U$ such that for a.e. $t \in (0,T)$,

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{G}\varepsilon(u(t)) - \mathcal{C}\theta(t) + \\ \int_0^t \mathcal{B}\left(\sigma(s) - \mathcal{A}\varepsilon(\dot{u}(s)) + \mathcal{C}\theta(s), \varepsilon(u(s))\right) ds \tag{2.35}$$

$$< \ddot{u}(t), w >_{\mathbb{V}',\mathbb{V}} + (\sigma(t), \varepsilon(w))_{\mathcal{H}} + \\ + j(\dot{u}(t), w) = < F(t), w >_{\mathbb{V}',\mathbb{V}}, \quad (\forall)\, w \in \mathbb{V} \tag{2.36}$$

$$(\dot{\theta}(t) + \mathcal{C} \cdot \nabla \dot{u}(t), \eta)_H + \\ + (K \nabla \theta(t), \nabla \eta)_{\mathbb{H}} + l(\theta(t), \eta) = \\ = (q(t), \eta)_{\mathbb{H}}, \quad (\forall)\, \eta \in U \tag{2.37}$$

$$u(0) = u_0 \; ; \; \dot{u}(0) = v_0 \; ; \; \theta(0) = \theta_0. \tag{2.38}$$

4. Existence and uniqueness of the solution

The main result of this section is the following theorem of existence and uniqueness of the weak solution in the thermomechanical problem **(P)**.

Theorem 3.1

Under the assumptions $(2.27) - (2.34)$, there *exists a unique solution* $\{\, u\, ;\, \sigma\, ;\, \theta \,\}$ of the *Problem* $(VP) \equiv (2.35) - (2.38)$. Moreover, the solution satisfies the regularity properties,

$$u \in W^{1,2}(0,T; \mathbb{V}) \cap C^1([0,T]; \mathbb{H}), \\ \ddot{u} \in L^2(0,T; \mathbb{V}'), \\ \sigma \in L^2(0,T; \mathcal{H}) ; \; div\, \sigma \in L^2(0,T; \mathbb{V}') \\ \theta \in L^2(0,T; U) \cap C^0([0,T]; H) ; \\ \dot{\theta} \in H^1(0,T; U'). \tag{3.1}$$

The proof of *Theorem 3.1* is based on the result concerning a fixed point strategy, similar to that used in [7], [3], [14]. It is carried out in several steps, and the variational problem has decomposed in three auxiliary problems meant to determine of the *displacement field*, the *stress field* and the *temperature field*, respectively.

5. Conclusions

Because of the importance of the continuous improvements to the casting processes of the steels, a considerable effort has been made in modeling and numerical simulations of the tribologycal contacts between casted products (slab, bloom, etc) and walls of the mould, and as well, between slab and the support (traction, soft-reduction) rolls, during the secondary cooling.

In the present paper has been investigated a mathematical model for *triboprocesses* involving the coupling *thermal* and *mechanical* aspects by specific behaviour laws of materials.

The dynamical contact has been described as the effect of a *normal* and *tangential damped response conditions*.

The *classical* as well as a *variational formulation* of the thermodynamical problem are presented.

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