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ASPECTS OF DYNAMIC MODELLING OF MOTOR VEHICLES WITH EMPHASIS ON THE INFLUENCE OF THE NONLINEAR CHARACTER OF ELASTIC FORCES IN TYRES

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ABSTRACT

The study of oscillatory motions in vehicles is of particular interest from the perspective of improving their dynamic performance. This aspect has been highlighted in the present work by emphasizing the scientific interest that has been identified in the specialized literature, with several representative studies brought to the forefront. The study proceeds with the presentation of a four-degree-of-freedom dynamic model capable of characterizing the oscillatory regime of a vehicle when passing over an obstacle. Based on this dynamic model, the kinematic parameters of the oscillatory motion of a vehicle as it passes over an obstacle were investigated, highlighting the influence of the nonlinear nature of the elastic forces in the tires on these parameters. The conclusions of the study indicate that accounting for the nonlinear characteristics of the tire elastic forces is necessary to achieve a more realistic characterization of the dynamic behavior of the analysed phenomenon; however, for the numerical values considered, the amplitudes of the displacements and velocities of the analysed masses do not differ significantly from those predicted by the linear model.

KEYWORDS: dynamics, vehicles, dynamic modeling, nonlinear elastic forces, obstacle crossing

1. Introduction

The issue of dynamic vehicle modeling during motion has attracted considerable interest in the scientific community, driven by efforts to improve comfort, maneuverability, and stability. In this regard, numerous research studies have extensively addressed this subject, with dynamic models continuously evolving in complexity to more accurately reflect real-world conditions.

In this context, Untaru and Tabacu (1981) proposed various dynamic models for characterizing the oscillatory motion of a vehicle: simple models with one degree of freedom, models with two degrees of freedom, and a more comprehensive four-degree-of-freedom model.

Jazar (2008) provides a classical approach to vehicle dynamics, presenting the general theory of oscillatory motions and emphasizing the fundamental mathematical models used to describe vehicle behavior. His work addresses essential aspects such as suspension modeling, the influence of unsprung masses, and the analysis of lateral stability.

Rajamani (2012) investigates control strategies aimed at improving vehicle dynamic performance, with a focus on advanced stability control and the optimization of tire–road interaction. The study highlights the importance of control algorithms and sensors in ensuring improved maneuverability, while underscoring the necessity of precise modeling of the mechanical systems involved.

Azadi, Vaziri, and Shahhoseini (2010) present a comprehensive model of a vehicle with a flexible body, simulated in MSC.ADAMS and MSC. NASTRAN for directional stability analysis. An optimized control strategy is proposed for the Vehicle Dynamics Control (VDC) system, combined with ABS, in order to maintain wheel slip within optimal limits. The comparative study between rigid and flexible models reveals significant differences in both vehicle dynamics and the control efforts required.

The present study analyses the dynamic behavior of a passenger car as it passes over

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obstacles, under the assumption that the vehicle can be modelled as a four-degree-of-freedom system (Fig. 1). Based on this model, vertical and pitch oscillations of the vehicle were studied comparatively, separately considering the cases in which tire elastic forces were modeled as linear or nonlinear.

2. Dynamic modeling of a vehicle passing over an obstacle

The dynamic study addressed in this work is based on the physical and mathematical models presented by Untaru and Tabacu (1981), which describe the vehicle as a rigid system with four degrees of freedom. Although this model includes four degrees of freedom, it introduces the assumption that the wheels of the same axle traverse an obstacle on the road surface simultaneously. In other words, the system is perturbed only when passing over obstacles whose length is at least equal to the width of the vehicle (Figure 1).

The mathematical model (Untaru & Tabacu, 1981) is a system of four second-order differential equations, given in equation (1).

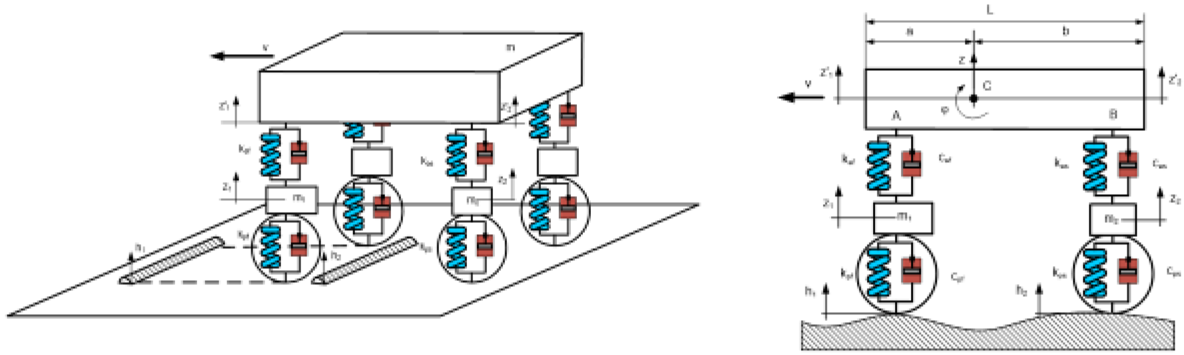

Fig. 1. *Physical model of vehicle with four degrees of freedom*

$$\begin{cases} m\cdot\ddot{z}+c_{af}\cdot\left(\dot{z}'_1-\dot{z}_1\right)+c_{as}\left(\dot{z}'_2-\dot{z}_2\right)+k_{af}\left(z'_1-z_1\right)+k_{as}\left(z'_2-z_2\right)=0 \\ m\cdot\rho^2\cdot\ddot{\varphi}+a\cdot c_{af}\cdot\left(\dot{z}'_1-\dot{z}_1\right)-b\cdot c_{as}\left(\dot{z}'_2-\dot{z}_2\right)+a\cdot k_{af}\left(z'_1-z_1\right)-b\cdot k_{as}\left(z'_2-z_2\right)=0 \\ m_1\cdot\ddot{z}_1+c_{af}\cdot\left(\dot{z}'_1-\dot{z}_1\right)-k_{af}\left(z'_1-z_1\right)+c_{pf}\left(\dot{z}_1-\dot{h}_1\right)+F_{nzf}\left.\right)=0 \\ m_2\cdot\ddot{z}_2-c_{as}\cdot\left(\dot{z}'_2-\dot{z}_2\right)-k_{as}\left(z'_2-z_2\right)+c_{ps}\left(\dot{z}_2-\dot{h}_2\right)+F_{nzs}\left.\right)=0 \end{cases} \qquad (1)$$

where the following notations are used:

m – suspended mass of the vehicle;

m_1, m_2 – unsprung masses at the front and rear, respectively;

z_1, z_2 – vertical displacements of the front (m_1) and rear (m_2) unsprung masses, respectively;

k_{af} – equivalent stiffness constant of the front axle springs;

k_{as} – equivalent stiffness constant of the rear axle springs;

k_{pf} – equivalent stiffness constant of the front axle tires;

k_{ps} – equivalent stiffness constant of the rear axle tires;

c_{af} – equivalent damping coefficient of the front axle springs;

c_{as} – equivalent damping coefficient of the rear axle springs;

c_{pf} – equivalent damping coefficient of the front axle tires;

c_{ps} – equivalent damping coefficient of the rear axle tires;

h_1, h_2 – heights of the road surface irregularities;

h – amplitude of the irregularities;

z_a, z_p – vertical displacements of the masses m_1 and m_2;

z'_a, z'_p – vertical displacements of points A and B;

z – displacement of mass m;

φ – angular displacement of the suspended mass;

a – distance from the centre of gravity to the front axle;

b – distance from the centre of gravity to the rear axle;

F_{nzf} – front axle nonlinear tire force;

F_{nzs} – rear axle nonlinear tire force.

and

$$F_{nz}f = k_{2f}(z_1-h_1)$$

$$F_{nz}s = k_{2s}(z_2-h_2)$$

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where:

k_{2f} – nonlinear stiffness coefficient of the front tire;

k_{2s} – nonlinear stiffness coefficient of the rear tire.

In the monograph by Jazar (2008), for the case of nonlinear elastic forces arising from tire deformation, it is assumed that this force is the superposition of two components, static and dynamic, as expressed in equation (2).

$$F_{nz} = F_{nzs} + F_{nzd} \qquad (2)$$

where:

F_{nz} – nonlinear elastic force due to vertical deformation z of the tires;

F_{nzs} – static component of the nonlinear elastic force;

F_{nzd} – dynamic component of the nonlinear elastic force.

For the model under consideration, the static and dynamic components of the nonlinear elastic tire force may be written analytically as shown in equations (3) and (4).

$$\begin{aligned} &F_{nzf}=F_{nzsf} + F_{nzdf} \\ &F_{nzsf} = \mathrm{k}_{1\mathrm{pfn}} \cdot (z_1 - h_1) + \mathrm{k}_{2\mathrm{pfn}} \cdot (z_1 - h_1)^2 \\ &F_{nzdf} = \mathrm{k}_{3\mathrm{pfn}} \cdot (\dot{z}_1 - \dot{h}_1) \\ &F_{nzf} = \mathrm{k}_{1\mathrm{pfn}} \cdot (z_1 - h_1) + \mathrm{k}_{2\mathrm{pfn}} \cdot (z_1 - h_1)^2 + \mathrm{k}_{3\mathrm{pfn}} \cdot (\dot{z}_1 - \dot{h}_1) \end{aligned} \qquad (3)$$

$$\begin{aligned} &F_{nzs}=F_{nzss} + F_{nzds} \\ &F_{nzss} = \mathrm{k}_{1\mathrm{psn}} \cdot (z_2 - h_2) + \mathrm{k}_{2\mathrm{psn}} \cdot (z_2 - h_2)^2 \\ &F_{nzds} = \mathrm{k}_{3\mathrm{psn}} \cdot (\dot{z}_2 - \dot{h}_2) \\ &F_{nzs} = \mathrm{k}_{1\mathrm{psn}} \cdot (z_2 - h_2) + \mathrm{k}_{2\mathrm{psn}} \cdot (z_2 - h_2)^2 + \mathrm{k}_{3\mathrm{psn}} \cdot (\dot{z}_2 - \dot{h}_2) \end{aligned} \qquad (4)$$

where:

F_{nzd} – static elastic force of nonlinear type in vertical direction for the tires of both axles;

F_{nzsf} – nonlinear static elastic force in the vertical direction for the front axle tires;

F_{nzsf} – nonlinear static elastic force in the vertical direction for the rear axle tires;

F_{nzsf} – nonlinear dynamic elastic force in the vertical direction for the front axle tires;

F_{nzds} – vertical direction nonlinear dynamic elastic force for rear axle tires;

k_{1pfn} – coefficient of nonlinear equivalent stiffness for front axle tires;

k_{1psn} – coefficient of nonlinear equivalent stiffness for rear axle tires;

k_{2pfn} – quadratic nonlinear equivalent stiffness coefficient of the front axle tires;

k_{2psn} – quadratic nonlinear equivalent stiffness coefficient of the rear axle tires;

k_{3pfn} – the equivalent stiffness coefficient for the dynamic component of the front axle tires;

k_{3psn} – the equivalent stiffness coefficient for the dynamic component of the rear axle tires.

According to the previously presented relations, the system of differential equations (1), which characterizes the considered physical model, can be written in the following form:

$$\begin{cases} m \cdot \ddot{z} + c_{1f} \cdot (\dot{z}'_1 - \dot{z}_1) + c_{1s}(\dot{z}'_2 - \dot{z}_2) + F_{nz} = 0 \\ m \cdot \rho^2 \cdot \ddot{\varphi} + a \cdot c_{1f} \cdot (\dot{z}'_1 - \dot{z}_1) - b \cdot c_{1s}(\dot{z}'_2 - \dot{z}_2) + a \cdot F_{nzsf} - b \cdot F_{nzss} = 0 \\ m_1 \cdot \ddot{z}_1 + c_{1f} \cdot (\dot{z}'_1 - \dot{z}_1) - F_{nzsf} + c_{2f}(\dot{z}_1 - \dot{h}_1) + k_{2f}(z_1 - h_1) = 0 \\ m_2 \cdot \ddot{z}_2 - c_{1s} \cdot (\dot{z}'_2 - \dot{z}_2) - F_{nzss} + c_{2s}(\dot{z}_2 - \dot{h}_2) + k_{2s}(z_2 - h_2) = 0 \end{cases} \qquad (5)$$

In classical dynamic models, the nonlinear elastic forces generated by tire deformation were often neglected to simplify the solution of systems of differential equations, due to limited computational resources. However, this simplification does not fully represent reality, as tires dissipate part of the energy resulting from deformation as heat.

3. Effect of Nonlinear Elastic Forces Tire

For the theoretical investigation of the influence of tire nonlinear behavior, it was assumed that the vehicle passes over an obstacle such as the one shown in Figure 1, at a speed of 50 km/h. Under this assumption, the dynamic excitation produced by the obstacle was modeled as a sinusoidal function (Figure 2).

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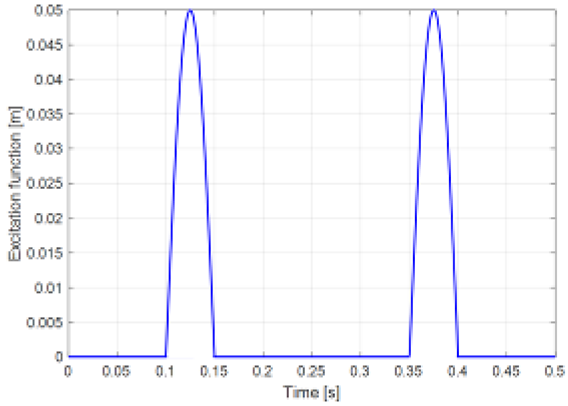

***Fig. 2.** Excitation function*

The solution of the system of differential equations was carried out using the numerical values from Table 1, corresponding to a road vehicle of category M1 The solution of the system of differential equations was carried out with the numerical values listed in Table 1, corresponding to a category M1 road vehicle.

***Table 1.** Numerical values of geometric, mass, and rheological parameters*

m, m_1, m_2	1500 kg, 70 kg, 70 kg
k_{af}, k_{as}, k_{pf}, k_{ps}	$25\cdot10^3$ N/m, $20\cdot10^3$ N/m, $200\cdot10^3$ N/m, $150\cdot10^3$ N/m
k_{1pfn}, k_{1psn} k_{2pfn}, k_{2pfn} k_{3psn}, k_{3pfn}	200000, 150000, N/m 100000, 80000, N/m^2 50000, 40000 Ns/m
c_{af}, c_{as}, c_{pf}, c_{ps}	2000 Ns/m, 2500 Ns/m, 2500 Ns/m, 2500 Ns/m
h_1, h_2	$h_1=h\cdot\sin(\omega\cdot t)$ m, h=0,1 m, ω=62,8 rad/s $h_2=h\cdot\sin(\omega\cdot t)$ m, ω=62,8 rad/s
a, b	1.07 m, 1.61 m

Using these numerical values, the system of second-order differential equations was solved in MATLAB R2021b, obtaining graphical representations of the kinematic parameters of the vehicle’s oscillatory motion for two scenarios:

- the elastic forces resulting from tire deformation are linear;
- the elastic forces resulting from tire deformation are nonlinear.

For ease of highlighting the differences between the two cases, the graphs of the analysed parameters have been presented in parallel.

4. Interpretation of the Results

For the masses m, m_1, and m_2, the variations of displacements and velocities were graphically represented in both the time and frequency domains, as well as in the phase plane, Figures 3-9.

In the time domain, the displacements and velocities of the suspended mass exhibit similar values for both models. The maximum oscillation amplitudes are not significantly affected by the inclusion of nonlinear terms in the tire model. In the phase plane, the resulting trajectories are nearly identical, reflecting system stability and the predominantly linear nature of the vehicle body’s response.

The spectral representations of the displacement indicate a clearly defined dominant frequency, without significant harmonics appearing in the nonlinear model. This suggests that, at the body level, the nonlinear characteristics of the tire forces have a limited effect under moderate dynamic conditions.

In both models, the trajectories of the suspended mass in the phase plane are elliptical and well-defined, indicating stable and lightly damped oscillations, characteristic of quasi-harmonic response systems. The differences between the linear and nonlinear trajectories are negligible, confirming the minor influence of tire nonlinearities on the vehicle body under the simulated conditions.

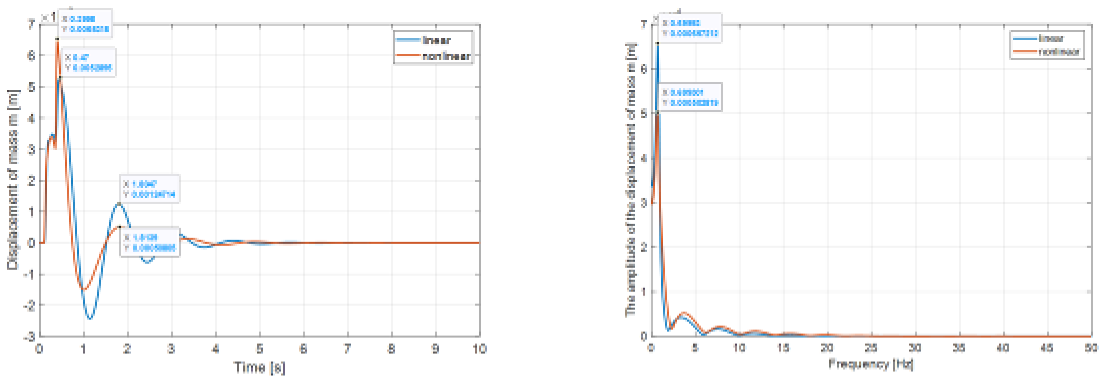

***Fig. 3.** Displacement of mass m [m]: left – linear; right nonlinear*

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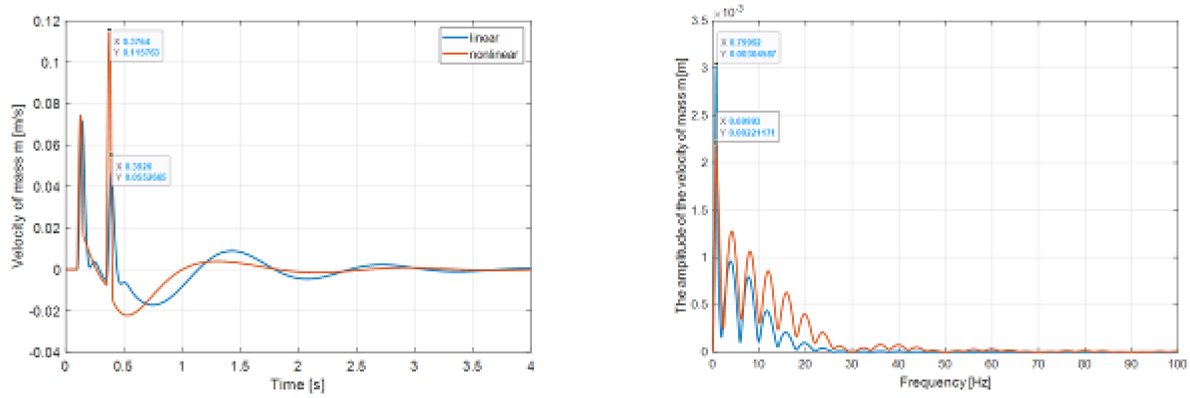

Fig. 4. *Velocity of mass m [m/s]: left – linear; right - nonlinear*

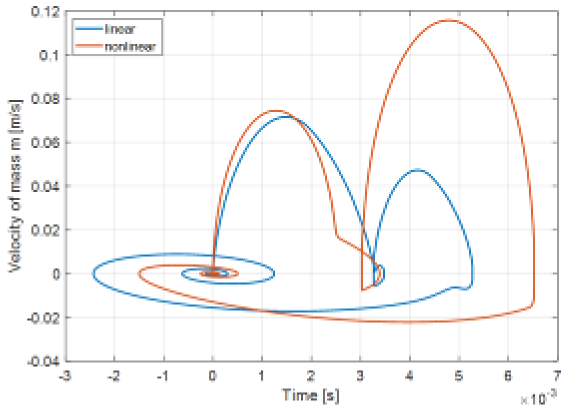

***Fig. 5**. Motion of mass m in the phase plane*

The front unsprung mass exhibits larger oscillations, being directly influenced by contact with surface irregularities and the rheological characteristics of the tires. In the case of nonlinear modeling, a slight attenuation of the peak amplitudes is observed, reflecting the capacity of the quadratic and dynamic terms to introduce additional energy dissipation.

The phase plane shows a slight dispersion of the trajectory in the nonlinear case, indicating more complex behavior with slightly increased variability in kinetic energy. The frequency spectrum reveals, in addition to the fundamental frequency, the appearance of higher-order harmonics in the nonlinear model, characteristic of the higher-order nonlinear effects of the tire–road system.

For mass m_1, the trajectories in the phase plane are wider and more asymmetric, reflecting more complex dynamic behavior. In the nonlinear model, a slight thickening of the trajectory is observed, signalling the presence of additional frequency components and an irregular damping effect. This phenomenon indicates a possible transition from harmonic oscillations to responses with multiple frequency content, typical of nonlinear systems.

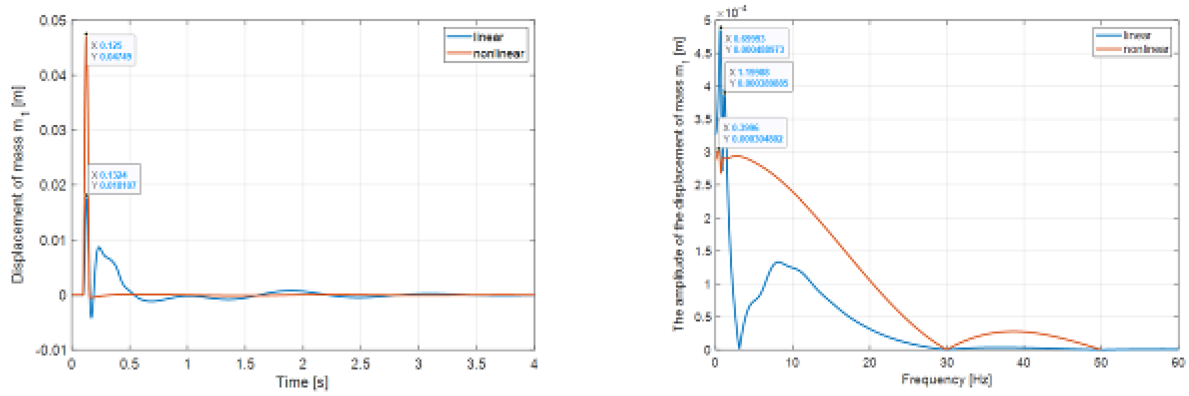

Fig. 6. *Displacement of mass m_1 [m]*

The behavior of the rear unsprung mass is similar to that of mass m_1, but with slightly smaller amplitudes due to the lower stiffness constants of the rear axle. A slight modification of the oscillation profile is observed in the nonlinear model, indicating a more damped response during the initial phase of motion.

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The phase-plane trajectories are more dispersed in the nonlinear model, particularly in the transient regime, suggesting more efficient energy dissipation. The frequency spectrum of the displacement of mass m_2 shows a slightly broader energy distribution in the nonlinear case, confirming the influence of the quadratic terms on the response spectrum.

Like mass m_1, the phase-plane trajectories of mass m_2 are slightly more dispersed in the nonlinear model, with a noticeable tendency for the curves to close more rapidly due to a better-damped response. This behavior confirms the role of different rheological coefficients on the rear axle and highlights the influence of nonlinearities in dissipating mechanical energy at the tire–bump contact.

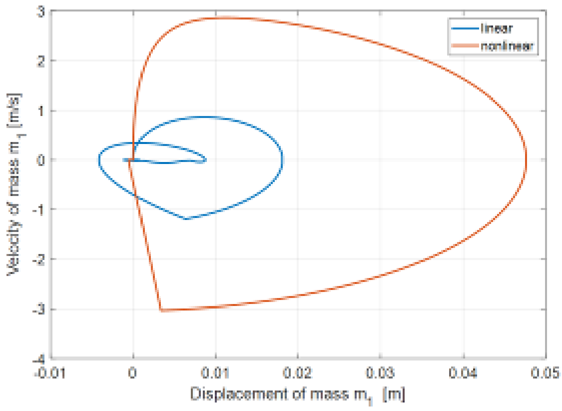

Fig. 7. *Motion of mass m_1 in the phase plane [m/s]*

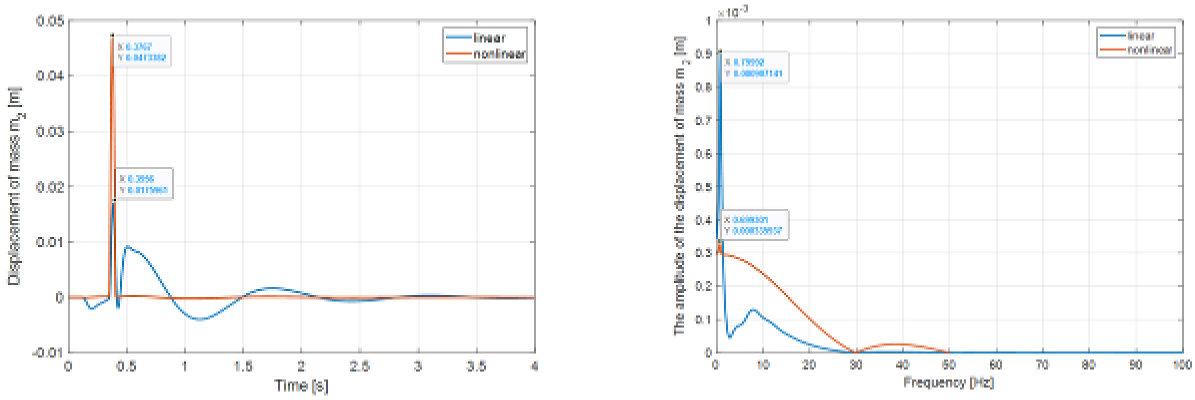

Fig. 8. *Displacement of mass m_2 [m]*

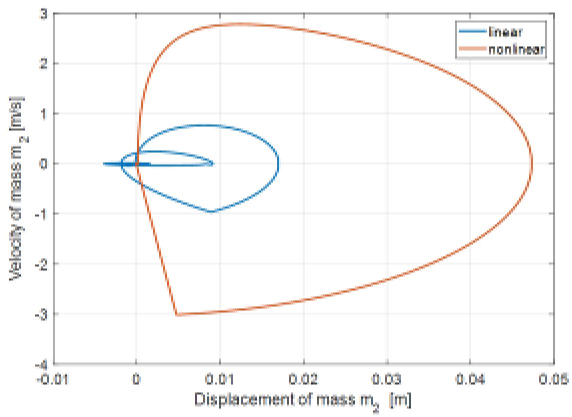

Fig. 9. *Motion of mass m_2 in the phase plane*

5. Conclusions

The present study analysed the oscillatory behavior of vehicle passing over road irregularities, using a four-degree-of-freedom model that considers both the sprung and unsprung components. The focus was placed on the influence of the nonlinear elastic forces developed by the tires compared to the classical linear model.

The results show that, under the simulated conditions, the amplitudes of the displacements and velocities of the analysed masses do not differ significantly between the two models. Nevertheless, nonlinear modeling is essential for realistically describing vehicle behavior under demanding dynamic conditions, where internal damping and energy dissipation effects become significant.

Moreover, the nonlinear formulation of the elastic forces allows for a natural extension of the study to transient simulations, testing advanced active suspension control strategies, as well as integrating complex tire–road interaction models.

As directions for future research, the development of a full spatial vehicle model is proposed, incorporating lateral and roll effects, as well as validating the theoretical model through experimental tests using sensors mounted on real vehicles, to correlate simulated results with real-world data.

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