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On the MAPLE Routines for Discrete Symmetries Study on Curved Manifolds

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ABSTRACT

The aim of this paper is to offer a tool to study the discrete symmetries for gauge minimally coupled charged spinless field to curved space-time. In this order, is made an algebraic description of basic discrete symmetries as space reversal P, time reversal T charge conjugation and their combination. We extend a previous MAPLE package procedures set in order to study the influence of the particular C, P, and T discrete transformation spinless field. In the last section, is proposed a simple method to find discrete symmetries of field differential equations. As an example of the application of the methods, we consider the discrete symmetries transformation of C, P, and T kind for a particular Gödel. Considering the Klein – Gordon – Maxwell – Einstein solutions, it is evaluated the electric current's components and further the boson system electric charge..
***Keywords**: Klein – Gordon field equation, Maxwell equation, discrette symmetry trasnformation.*

1. Introduction

Symmetry is the driving concept in particle physics. In Quantum Field Theory the particles are defined as finite dimensional irreps of the space-time and internal symmetry groups. In Statistical Mechanics the notion of symmetry has also played a very important role in the past, as a way of characterizing degrees of freedom and types of interaction.

In the last three decades, field theories on curved manifolds with significant applications to Cosmology have been intensively investigated leading to various exciting results that shad quite a new light on our understanding of the Universe [1].

The study of boson stars (BS) started with the work of Kaup [1] and Ruffini and Bonazzalo [2], who found asymptotically solutions of the Einstein-Klein-Gordon equations for spherically symmetric equilibrium state.

The presence of dark matter has been established indirectly in a wide range of scale of the universe, from that of individual galaxies to the entire universe itself. Though direct measurements of the nature of the dark matter have not yielded any result, speculations on its composition vary from baryonic to non-baryonic matter.

Since the general-relativistic analytical study of the coupled field equations is of a real interest for a better understanding of different stellar configurations as well as for a numerical-functional combined iterative treatment which describes the dynamics of charged boson nebulae, a MAPLE package procedure set is welcome. The huge volume of computations and the necessity of checking the result were the main reason for a software algorithm.

In the first section is revealed the MAPLE algorithm in order to underline the discrete symmetry transformations procedures.

2. Fields equations on curved space-time

Considering a complex scalar field minimally coupled to a spherically symmetric space – time, in order to derive the Klein--Gordon--Maxwell--Einstein system of equations should employee a pseudo-orthonormal tetradic frame, $\{e_a\}_{a=\overline{1,4}}$, in order to have a Minkowskian metric tensor

$$\eta_{ab} = diag[1,1,1,-1] \quad (1)$$

For a charged massive boson, coupled to the electromagnetic field, the system is described by a Lagrangian density of the form

$$L = L(\Phi, \overline{\Phi}, \Phi_{,b}, \overline{\Phi}_{,a}) + \frac{1}{4}F^{ab}F_{ab} \quad (2)$$

where the gauge covariant - read

$$\Phi_{,a} = \Phi_{|a} - ieA_a\Phi$$

and respectively

$$\overline{\Phi}_{,a} = \overline{\Phi}_{|a} + ieA_a\overline{\Phi} \quad (3)$$

The Maxwell tensor

$$F_{ab} = A_{b:a} - A_{a:b} \quad (4)$$

is expressed in the terms of the Levi-Civita covariant derivative of the four-potential $\{A_a\}_{a=\overline{1,4}}$, i.e.

$$A_{a;b} = A_{a|b} - A_c\Gamma^c{}_{ab} \quad (5)$$

The symbols $\Gamma^c{}_{ab}$ are defined as

$$\Gamma^c{}_{ab} = g^{dc}\frac{1}{2}(g_{ad,b} + g_{db,a} - g_{ab,d})$$

and represents the Christoffel symbols.

By varying with respect to different fields, we come to the Klein--Gordon--Maxwell (KGM) system of equations:

Building up the energy-momentum tensor [1, 2, 6, 9] it can be derived the Einstein equation

$$G_{ab} = kT_{ab} \quad , \quad (6)$$

where the tensor have the explicit form as.

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}, \quad (7)$$

3. MAPLE procedures

In order to study these discrete symmetry transformations, we extend a previous MAPLE packages [3, 5]. The main aim was to implement the algebraic description of basic discrete symmetries as space reversal P, time reversal T charge conjugation and their combination proprieties on the interacting fields' systems study. In this way, it had to use our previous software approaching in order to succeed in writing down the Klein--Gordon—Maxwell - Einstein system equation. In this section we shall describe only the additional software procedure parts.

The first step of the program starts after initializing the main used package, with a set of definitions for the entire set of necessary objects.

In the second level, it should be set the considered transformations parameters- charge conjugation, space reversal or temporal inversion or their combinations. This stage is necessary on order to minimize the amounth of used memory. In this way, the algorithm will run, following the same steps [2, 3], and, at the finish stage, could be made comparisons for computed magnitudes.

The kernel of considered discrete symmetries have the form

```
> coord:=[        ];
> TransT_Polar:=proc(ax)
      tr1:={coord[1]=xi,   coord[2]=psi,
coord[3]=zeta, coord[4]=-tau}:
      dchange(tr1,ax);
  end proc;

> Trans_identic:=proc(ax)

>tr1:={xi=coord[1],psi=coord[2],zeta=c
oord[3],tau=coord[4]}:
      dchange(tr1,ax)
  end proc;

> SimT_Polar:=proc(ax) :
      eq1:=TransT_Polar(ax):
       Trans_identic(eq1):
  end proc;
```

In the third stage, using the MAPLE platform, should be compute the needed tranformed geometry elements as the first and second order derivatives of the metric tensor , the Christoffel symbols, , the Riemann and Ricci tensors and finally, the Einstein tensor components:

As in our previous works, it was introduced a pseudo-orthonormal tetradic frame, with the metric tensor of Minkowskian space-time, (1). The transformation tensors and respectively, defined as in our previous papers [2, 5, 8].

Considering a complex scalar field minimally coupled to a spherically symmetric space – time, In order to study the proprieties of these kind of interacting fields, was considered the necessity of building a helpful software in order to succeed in writing down the Klein--Gordon—Maxwell system equation and to study the C, P and T transformation. In this section we shall describe the structure and the main features of the procedures of the formalism.

The first part of the program starts after initializing the main used package, with a set of definitions for the entire set of necessary objects. For example, using the MAPLE platform, one can write (for the particular case of using Cartesian coordinates)

>coord:=[x,y,z,t];

>Mg:= array (1 .. 4 , 1 ..4 , symmetric, [(1,1)=g11(coord[1], ….coord[4])),

(1,2) =g12 (coord [1] , coord[2] , coord[3] , coord[4]),….,

…………………

(4,4)=g44(coord[1],coord[2],coord[3],coord[4])])

>g:=create([-1,-1],op(Mg));

In this stage it could be set the considered transformations - charge, parity or temporal inversion. The algorithm will run, following the same steps, and, at the finish stage, could be made comparisons for computed magnitudes.

In the second stage, using the MAPLE platform, should be compute the needed geometry elements as the first and second order derivatives of the metric tensor , the Christoffel symbols, , the Riemann and Ricci tensors and finally, the Einstein tensor components, as it follows:

> D1g:=d1metric(g,coord);

> D2g:=d2metric(D1g,coord);

> Cf1:=Christoffel1(D1g);

…………………………

*> Einstein1:=lin_com(1,Rc,1/2*Rsc,g);*

In order to get a simpler set of differential equations, it is more convenient to introduce a pseudo-orthonormal tetradic frame, with the metric tensor of Minkowskian spacetime, (1). The transformation tensors and respectively , defined in the following command lines, are needed to turn to the tetradic frame.

> Mgc : = array (1..4 , 1..4 , symmetric,[(1,1)=1,(1,2)=0……. (4,4)= -1]);

> g_c:=create([-1,-1],op(Mgc));

> frame(g_c, h1inv, const_g, coord);

> eval(h1_inv);

…………………………

> change_basis(g_c,h1,h1inv);

The definition of the gauge-covariant derivatives of the complex scalar field, (3), is an important step in building the system lagrangian and in writing down the KGM coupled equations. In order to get a coherent structure, the terms are defined as tensors. The main advantage of this form is the simplicity on the transformation between the two considered frames. The gauge fields $\{A_a\}_{a=\overline{1,4}}$ can be defined in a direct manner and, using the tensor structure of definition, it can be easily used in building the gauge derivatives of the scalar field.

> P:=create ([], Phi (coord[1], coord[2], coord[3], coord[4]));

> P_conjugate := create([] , conjugate (Phi(coord[1],….,coord[4])));

> P_bar_a:=partial_diff(P,coord);

…………………………

> A_gauge:=create([-1],array(1..4, [(1)=A[1](coord[1],……coord[4]),

(2)=A[2](coord[1],coord[2],coord[3],coord[4]),

(3)=A[3](coord[1],coord[2],coord[3],coord[4]),…….

> P_bar_a_2:=prod(A_gauge,P);

*> P_coma_a:=lin_com(1,P_bar_a,-I*E,P_bar_a_2);*

> P_bar_a_conjugate : = prod(P_conjugate,A_gauge);

…………………………

Further, using a tensor for the interaction term in the Klein - Gordon equations, the evolution equation for the complex scalar field Φ reads:

> prod(A_gauge,P_bar_a_conjugate);

> J_Klein_Gordon1 :=prod (A_gauge_up, P_bar_a, [1,1]);

> J_Klein_Gordon_temp :=prod (A_gauge,P);

> J_Klein_Gordon2 := prod (A_gauge_up, J_Klein_Gordon_temp,[1,1]);

> ……..

> Klein_Gordon1 :=prod (g_inv, P_bar_a, [2,1]);

> Klein_Gordon2 :=partial_diff (Klein_Gordon1 ,coord);

> Klein_Gordon3 :=contract (Klein_Gordon2,[1,2]);

> Klein_Gordon_kinetic := lin_com (1/det_g,Klein_Gordon3);

> Klein_Gordon5: =prod(P_conjugate,P);

> ……..

> Eq_Klein_Gordon := Klein_GordonM=J_Klein_GordonM;

The next step is devoted to the Maxwell tensor F_{ab}, defined in (4), and to the corresponding Maxwell equations (7). The currents and the interaction term are put in the same tensorial structure.

> Fab1:=cov_diff(A_gauge, coord,Cf2);

> Fba2:=permute_indices(Fab1,[2,1]);

> Fab:=lin_com(1,Fab1,-1,Fba2);

> ……

> JMaxwell2: =prod (P, P_coma_a_conjugate);

> JMaxwell3: =lin_com(1,JMaxwell1,-1,JMaxwell2);

```
> JMaxwell4: =prod (g_inv,JMaxwell3,[2,1]);
> JMaxwell: =lin_com(-I*E,JMaxwell4);
………………….
```

One may notice that these steps are sufficient to build the Maxwell equations and, by using the tensors: h_1 whose character is [+1,-1] and his invert - h_{1inv} , one can turn to the initial frame system.

```
> Fab_b:=cov_diff(Fab,coord,Cf2);
> Fab_b1:=partial_diff(Fab_up,coord);
> Fab_b2:=prod(Fab_up,Cf2,[1,2]);
> ……;
> JMaxwellM:=JMaxwell[compts];
> for i from 1 to 4 do eqMaxwell[i]:=MaxwellM[i]=JMaxwellM[i] end do;
```

At this point, an important achievement is the building of necessary Lorentz condition. In order to introduce the whole lagrangian of the system, given by (2), and to build the energy--momentum tensor involved in the Einstein equations one needs the following procedure part. This can generate a rank zero tensor-type object which can be employed in writing down the energy-momentum tensor, T_{ab} , given by (10):

```
> L1:= prod (P_coma_a_conjugate , P_coma_a);
> Lkinetic:= prod(g_inv,L1,[1,1],[2,2]);
> L2:=prod (P_conjugate,P);
> Lmass:= lin_com(m0^2,L2);
> …….;
> Tab:= lin_com(1,Tab1,1,Tba1,1,Tab2,-1,Tab3);
> J_Einstein:=l in_com(Kappa,Tab);
```

Putting together (9) and the objects defined in previous steps of the algorithm, the Einstein equations (8) explicitly are:

```
> EinsteinM:=Einstein1[compts];
> J_EinsteinM:=J_Einstein[compts];
> for i from 1 to 4 do : for j from 1 to 4 do
   EqEinstein[i,j ]:= EinsteinM[i,j] = J_EinsteinM[i,j] end do : end do;
> …….
```

Finally, one has to impose specific ansatz conditions, in order to obtain a set of simpler equations, easier to manipulate and of course, much interesting for didactic reasons.

In the last part of this short presentation, will be briefly discussed only procedures used in order to obtain a first order perturbative solution for the coupled field equations' system. This approach starts with the physically reasonable assumption that the charged scalar field is the main source of both the electromagnetic and gravitational fields. Considering it, in the first instance, could be neglected the feedbacks of gravity and electromagnetism on the charged scalar source [2, 6].

To succeed in this purpose, it has to build field equations in an Euclidean approximation, using null Christoffel symbol values for the field covariant derivative and null metric tensor functions. In a coherent approaching, it should be computed all the necessary elements for the Klein - Gordon - Maxwell system equations.

However, in the most studied cases, for an analytical approaching, can be obtained only approximate solutions of different rank.

4. Specific results

Let us consider a spherically symmetric configuration describe by a metric tensor of static conformal type, expressed in Schwarzschild coordinates as

$$ds^2 = e^{2(H(r))}dr^2 + r^2 d\theta^2 + \\ + e^{2(G(r))}d\varphi^2 - e^{2(F(r))}dt^2 \qquad (7)$$

The Christoffel symbols derived in this frame are

$$\Gamma^2{}_{12} = -\Gamma^1{}_{22} = \frac{1}{r}e^{-(H(r))}$$
$$\Gamma^3{}_{13} = -\Gamma^1{}_{33} = G'(r)e^{-(H(r))}$$
$$\Gamma^4{}_{14} = \Gamma^1{}_{44} = F'(r)e^{-(H(r))} \qquad (8)$$

where we used

$$F'(r) = \frac{dF(r)}{dr} \quad \text{and} \quad G'(r) = \frac{dG(r)}{dr}$$

The Einstein tensor G_{ab} has the following non-vanishing components

$$G_{11} = \\ = \frac{1}{r}\left[G'(r) + F'(r) + rF'(r)G'(r)\right]e^{-2(H(r))}$$

$$G_{22} = \\ = \left[G''(r) + G'(r)^2 - H'(r)G'(r) + \\ + F''(r) + F'(r)^2 - H'(r)F'(r) + \\ + F'(r)G'(r)\right]e^{-2(H(r))}$$

$$G_{33} = \frac{e^{-2(H(r))}}{r}\left[-H'(r) + rF''(r)^2 - \\ - rH'(r)F'(r) + F'(r) + rF'(r)^2\right]$$

$$G_{44} = \frac{e^{-2(H(r))}}{r}\left[-H'(r) + rG''(r) + \right. \qquad (9)$$
$$\left. + rG'(r)^2 - rH'(r)G'(r) + G'(r)\right]$$

The Maxwell tensor

$$F_{ab} = A_{b:a} - A_{a:b}$$

in the particular case of working in the minimally symmetric ansatz $A_1 = A_1(r,t)$, $A_4 = A_4(r,t)$, $\Phi = \Phi(r,t)$, has a single non-vanishing Maxwell tensor (9) component is

$$F_{14} = -F_{41} =$$
$$= -\frac{\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]}{e^{H(r)}e^{F(r)}} \qquad (10)$$

Building up the energy-momentum tensor

$$T_{ab} = \overline{\Phi}_{:a}\Phi_{:b} + \overline{\Phi}_{:b}\Phi_{:a} + \qquad (11)$$
$$+ F_{ac}F_b^{\ c} - \eta_{ab}L$$

where the energy-momentum tensor T_{ab} has the explicit form

$$T_{11} = -m_0^2\overline{\Phi}\Phi + e^2\overline{\Phi}\Phi\left(A_1^{\ 2} + A_4^{\ 2}\right) +$$
$$+ ie\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) +$$
$$ie\frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right) +$$
$$+\left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} + \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] +$$
$$-\frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}} \times$$
$$\times\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2$$

$$T_{33} = T_{22}$$
$$T_{22} = -m_0^2\overline{\Phi}\Phi - e^2\overline{\Phi}\Phi\left(A_1^{\ 2} - A_4^{\ 2}\right)$$
$$-\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) +$$
$$+\frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right) +$$
$$-\left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} - \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] +$$
$$-\frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}} \times$$
$$\times\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2$$

$$T_{44} = m_0^2\overline{\Phi}\Phi + +e^2\overline{\Phi}\Phi\left(A_1^{\ 2} + A_4^{\ 2}\right) +$$
$$+\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) +$$
$$+\frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right) +$$
$$+\left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} + \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] +$$
$$+\frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}} \times$$
$$\times\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2$$

The explicit form for the Klein - Gordon equation is

$$e^{-2(H(r))} \times$$
$$\times\left[\Phi_{,r}\left(\frac{1}{r} - H'(r) + G'(r) + F'(r)\right) + \Phi_{,rr}\right] -$$
$$-e^{-2(F(r))}\Phi_{,tt} - m_0^{\ 2}\Phi = \qquad (12)$$
$$= 2ie\left\{e^{-(H(r))}A_1\Phi_{,r} - e^{-(F(r))}A_4\Phi_{,t}\right\} +$$
$$+ e^2\Phi\left[(A_1)^2 - (A_4)^2\right]$$

Unde the time reversal and under parity transformation the left part of the Klein – Gordon equation is invariant while the right side is not. For a good corelations, it should be underlined that, under a combined transpfrmation as time, parity and charge conjugation, the Klein – Gordon and Maxwell equation are invariant.

5. Aknowlegment

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MAPLE software und ablösbar krumm Raum unauffällig Veränderung

Die Article Absicht ist anbieten ein sher gut Gerät ablösbar die unauffällig Veränderung. Hier ist ein Computerprogram vermehrt und es mochte aufrufen und aufbessern der unserer dreite Computer program. Der Ausgang ist geprüft nur einen klassisch Einzelfall. Die Verhältnisen Musterung Feld sind zeit.

Algoritm pentru profilarea corectivă a sculelor pieptene

Scopul lucrării este de a oferi un instrument pentru studiul simetriilor discrete ale spațiilor curbe. În acest sens este realizată o descriere algebrică a simetriilor discrete de bază ce inversează spațiul (p), a timpului (T), compunerea sarcinilor şi combinarea acestora. Se realizează extinderea procedurilor pachetelor MAPLE anterioare. În final se propune o metodă simplă pentru determinarea simetriilor discrete ale ecuațiilor diferențiale ale câmpurilor.

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