3-D MODEL OF THE POTENTIAL FLOW PAST AN IMMERSED BODY

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ABSTRACT

The paper describes a method of computation for the 3-D potential flow past an immersed body. The hidrodynamic model is obtained by superposing a horizontal stream and a vertical stream over the flow generated by more vertical surfaces of linear sources yielding in horizontal planes and set in the vertical symetry plane of the immersed body. Then the model is applied for a 3D potential flow over an ellipsoid.

Keywords: potential flow, linear source, vertical surface of linear sources, 3D model

1. The Theoretical Model

Let us consider a vertical surface of linear sources set between the abscisaes *a* and *b* in the vertical plane *Oxz* of the Cartezian *Oxyz* system (see Fig.1). Each linear source yields in a horizontal plane and has a variable flow rate depending on an unknown law $\frac{q}{2\pi}=\frac{q(z)}{2\pi}$, where *q*(*z*) is the specific flow rate(on unit of length).

We consider the next complex potential in a horizontal plane at a certain level *z* [1] :

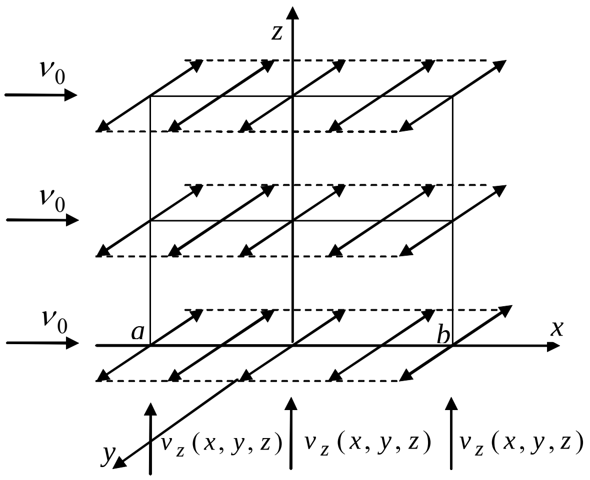

Fig. 1

$$W(\rho)=\frac{q(z)}{2\pi}\left[(a-b)+(\rho-a)\ln(\rho-a)+(\rho-b)\ln(\rho-b)\right] \quad (1)$$

where $\rho = x+iy, \left(i=\sqrt{-1}\right)$

The components v_x and v_y of the fluid velocity at a certain *z*-level are:

$$v_x=\frac{q(z)}{2\pi}\ln\frac{r'}{r''}$$
$$v_y=-\frac{q(z)}{2\pi}\left(\theta'-\theta''\right) \quad (2)$$

where

$$r'=|\rho-a|=\sqrt{(x-a)^2+y^2}$$
$$\sin\theta'=\frac{y}{r'} \quad , \quad \cos\theta'=\frac{x-a}{r'}$$
$$r''=|\rho-b|=\sqrt{(x-b)^2+y^2} \quad (3)$$
$$\sin\theta''=\frac{y}{r''} \quad , \quad \cos\theta''=\frac{x-b}{r''}$$

We superpose over the vertical surface of linear sources a vertical stream along *Oz* direction (set in the *xOz* plane), for the time being unknown, having a velocity $v_z=v_z(x,y,z)$ and an axial stream along *Ox* direction (also set in the *xOz* plane), having a constant velocity v_0.

The resulted velocity field is [1]:

$$v_x = v_0 + \frac{q(z)}{2\pi}\ln\sqrt{\frac{(x-a)^2+y^2}{(x-b)^2+y^2}}$$

$$v_y = -\frac{q(z)}{2\pi}\left[\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \arcsin\frac{y}{\sqrt{(x-b)^2+y^2}}\right] \tag{4}$$

$$v_z = v_z(x,y,z)$$

Considering an incompressible fluid and a potential flow, the velocity field has to be solenoidal and irrotational , this means it has to satisfy the equations:

$$\nabla \bar{v} = 0 \tag{5}$$

$$\nabla \times \bar{v} = 0 \tag{6}$$

By replacing eq.(4) in eq.(5) we obtain:

$$\frac{\partial}{\partial x}\left[v_0 + \frac{q(z)}{2\pi}\ln\sqrt{\frac{(x-a)^2+y^2}{(x-b)^2+y^2}}\right] - \\ -\frac{\partial}{\partial y}\left[\frac{q(z)}{2\pi}\left(\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \arcsin\frac{y}{\sqrt{(x-b)^2+y^2}}\right)\right] + \frac{\partial v_z}{\partial z} = 0 \tag{7}$$

or, if working out the partial derivatives,

$$\frac{q(z)}{2\pi}\left[\frac{|x-a|}{(x-a)^2+y^2} - \frac{|x-b|}{(x-b)^2+y^2} - \frac{|x-a|}{(x-a)^2+y^2} + \frac{|x-b|}{(x-b)^2+y^2}\right] + \frac{\partial v_z}{\partial z} = 0,$$

hence,

$$\frac{\partial v_z}{\partial z} = 0,$$

This means,

$$v_z = v_z(x,y,z) \tag{8}$$

By replacing eq.(4) in eq.(6) we obtain :

$$\frac{\partial v_z}{\partial y} = -\frac{\partial}{\partial z}\left[\frac{q(z)}{2\pi}\left(\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \arcsin\frac{y}{\sqrt{(x-b)^2+y^2}}\right)\right] \tag{9}$$

$$\frac{\partial v_z}{\partial x} = \frac{\partial}{\partial z}\left[v_0 + \frac{q(z)}{2\pi}\ln\sqrt{\frac{(x-a)^2+y^2}{(x-b)^2+y^2}}\right] \tag{10}$$

$$-\frac{\partial}{\partial x}\left[\frac{q(z)}{2\pi}\left(\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \arcsin\frac{y}{\sqrt{(x-b)^2+y^2}}\right)\right] = \\ = \frac{\partial}{\partial x}\left[v_0 + \frac{q(z)}{2\pi}\ln\sqrt{\frac{(x-a)^2+y^2}{(x-b)^2+y^2}}\right] \tag{11}$$

The satisfying of eq.(11) is to stated at once and eq.(9) and (10) become

$$\frac{\partial v_z}{\partial y} = -\frac{1}{2\pi}\left[\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \arcsin\frac{y}{\sqrt{(x-b)^2+y^2}}\right]\frac{dq(z)}{dz} \tag{9'}$$

$$\frac{\partial v_z}{\partial x} = \frac{1}{2\pi} \ln \sqrt{\frac{(x-a)^2 + y^2}{(x-b)^2 + y^2}} \frac{dq(z)}{dz} \tag{10'}$$

By partially integrating the eq.(9') in respect of y, we obtain:

$$v_z = -\frac{1}{2\pi}\frac{dq(z)}{dz}\left[y \arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \frac{|x-a|}{2}\ln\left((x-a)^2+y^2\right) - y\arcsin\frac{y}{\sqrt{(x-b)^2+y^2}} + \frac{|x-b|}{2}\ln\left((x-b)^2+y^2\right)\right] + C_1^{'}(x,z)$$

(12)

The function $C_1^{'}(x,z)$ can be found by replancing the velocity v_z as given by eq.(12)into eq.(10').

Thus

$$-\frac{1}{2\pi}\frac{dq(z)}{dz}\left[\frac{-y^2}{(x-a)^2+y^2} - \frac{1}{2}\ln\left((x-a)^2+y^2\right) - \frac{(x-a)^2}{(x-a)^2+y^2} + \frac{y^2}{(x-b)^2+y^2} + \frac{1}{2}\ln\left((x-b)^2+y^2\right) + \frac{(x-b)^2}{(x-b)^2+y}\right] =$$

$$= \frac{1}{2\pi}\frac{dq(z)}{dz}\ln\sqrt{\frac{(x-a)^2+y^2}{(x-b)^2+y^2}}$$

Hence,

$$\frac{\partial C_1^{'}}{\partial x} = 0,$$

or

$$C_1^{'} = C_1^{'}(z) \tag{13}$$

There results the following expression for the velocity v_z from eq.(12) and (13) :

$$v_z = -\frac{1}{2\pi}\frac{dq(z)}{dz}\left[y\cdot\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \frac{|x-a|}{2}\ln\left((x-a)^2+y^2\right) - y\cdot\arcsin\frac{y}{\sqrt{(x-b)^2+y^2}} + \frac{|x-b|}{2}\ln\left((x-b)^2+y^2\right)\right] + C_1^{'}(z) \tag{14}$$

There is to be notice that eq.(14) reduces to the form

$$v_z = C_1^{'}(z) \tag{15}$$

for $a=b$.

As v_z doesn't depend on z (because $\partial v_z / \partial z = 0$), we deduce that

$$C_1^{'}(z) = C_1^{'} = const \tag{16}$$

and so eq.(14) can be reduced to the form :

$$v_z = -\frac{1}{2\pi}\frac{dq(z)}{dz}\left[y\cdot\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \frac{|x-a|}{2}\ln\left((x-a)^2+y^2\right) - y\cdot\arcsin\frac{y}{\sqrt{(x-b)^2+y^2}} + \frac{|x-b|}{2}\ln\left((x-b)^2+y^2\right)\right] + C_1^{'} \tag{17}$$

Differentiating eq.(17) in respect of z and taking into account that $\partial v_z / \partial z = 0$, we obtain:

$$\frac{d^2 q(z)}{dz^2} = 0 \tag{18}$$

equivalent to

$$q(z) = C_2 z + C_3 \tag{19}$$

Considering eq.(19), eq.(17) will be rewritten under the form

$$v_z = -\frac{C_2}{2\pi}\left[y \arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \frac{|x-a|}{2}\ln\left((x-a)^2+y^2\right) - y\arcsin\frac{y}{\sqrt{(x-b)^2+y^2}} + \frac{|x-b|}{2}\ln\left((x-b)^2+y^2\right)\right] + C_1^{'} \quad (20)$$

Considering eq.(19) and (20), we can rewrite eq.(4) as

$$\begin{aligned} v_x &= \frac{C_2 z + C_3}{2\pi}\ln\sqrt{\frac{(x-a)^2+y^2}{(x-b)^2+y^2}} \\ v_y &= -\frac{C_2 z + C_3}{2\pi}\left[\arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \arcsin\frac{y}{\sqrt{(x-b)^2+y^2}}\right] \\ v_z &= -\frac{C_2}{2\pi}\left[y \arcsin\frac{y}{\sqrt{(x-a)^2+y^2}} - \frac{|x-a|}{2}\ln\left((x-a)^2+y^2\right) - y\arcsin\frac{y}{\sqrt{(x-b)^2+y^2}} + \frac{|x-b|}{2}\ln\left((x-b)^2+y^2\right)\right] + C_1^{'} \end{aligned} \quad (21)$$

2. The Formulation of the 3-D Potential Flow past an Immersed Body

We consider the above mentioned model for the computation of incompressible three-dimensional potential flow past an immersed ellipsoid.

Let *n* vertical surfaces of linear sources, of unknown specific flow rates $Q_i = C_{2i} z + C_{3i}$, *i*=1,2,..,*n*, yielding in horizontal planes and set in the vertical simetry plane of an ellipsoid (see Fig.2)

For the computation of the velocity in a certain point of the surfaces of the ellipsoid, considering the Cartesian system from Fig.2 *x* becomes $x - \xi_i$ and eq.(21) becomes:

$$\begin{aligned} v_x &= v_0 + \sum_{i=1}^{n}\left[\frac{C_{2i} z + C_{3i}}{2\pi}\ln\sqrt{\frac{(x-\xi_{i-1})^2+y^2}{(x-\xi_{i-1})^2+y^2}}\right] \\ v_y &= -\sum_{i=1}^{n}\left\{\frac{C_{2i} z + C_{3i}}{2\pi}\left[\arcsin\frac{y}{\sqrt{(x-\xi_{i-1})^2+y^2}} - \right.\right. \\ &\left.\left. - \arcsin\frac{y}{\sqrt{(x-\xi_i)^2}}\right]\right\} \\ v_z &= -\sum_{i=1}^{n}\left\{\frac{C_{2i}}{2\pi}\left[y\arcsin\frac{y}{\sqrt{(x-\xi_{i-1})^2+y^2}} - \right.\right. \\ &- \frac{|x-\xi_{i-1}|}{2}\ln\left((x-\xi_{i-1})^2+y^2\right) + \\ &+ y\arcsin\frac{y}{\sqrt{(x-\xi_i)^2+y^2}} \\ &\left.\left. + \frac{|x-\xi_i|}{2}\ln\left((x-\xi_i)^2+y^2\right)\right]\right\} + C_1 \end{aligned} \quad (22)$$

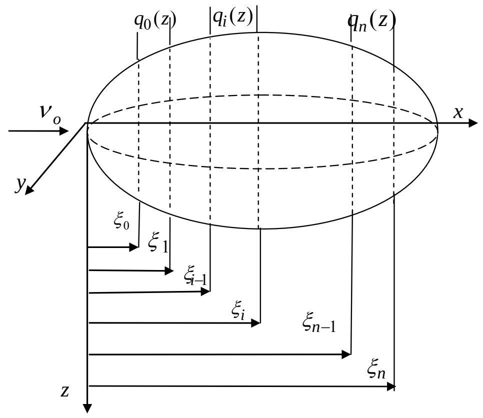

Fig. 2

where $C_1 = \sum_{i=1}^{n} C_{1i}^{'}$ can be taken as a unique constant.

There are 2n+1 unknowns C_{2i}, C_{3i}, (i=1,2,..n) and C_1. In order to fiind these unknowns which give the law of variation for the specific flow rates $q_i = C_{2i} + C_{3i}$, we consider the condition $\bar{n}_j \bar{v}_j = 0$, j=1,2,..,p, where p is the complete number of points which numerically defines the surface of the ellipsoid.

There results a linear system of p equation with 2*n*+1 unknowns. To obtain a determined system we choose the number of vertical sources with respect to the condition *p=2n+1* (*p* must be odd) . Hence, *n=(p-1)/2*.

REFERENCES

[1] Munson R. Bruce, Young F. Donald, *Fundamentals of Fluid Dynamics*, Wiley & Sons, Inc., Fifth Ed., 2006.

[2] Andrei, V., Popescu, F., *Hidrodynamic Model for Determining the Pressure and Velocity Field for the Potential Flow Around a Given Profile*, The Annals of „Dunărea de Jos University of Galați, Fasc.II, pag.12-16, 1994.