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ANALYTICAL AND FEM METHODS FOR STRESS STATE ESTIMATION IN A SHEARED PLANE WITH TWO IDENTICAL CIRCULAR HOLES

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ABSTRACT

Shear stresses occur in almost any applications during machining of mechanical parts. Inhomogeneities perturb the stress field in working pieces, meaning that they are responsible of a strong stress gradient and therefore cracks may occur in the regions in the vicinity of these inhomogeneities. The present paper presents some results concerning the stress state for a plane with two circular identical holes, subjected to pure shearing. To find the stress state the authors used analytical and numerical methods, on the basis of theory of elasticity hypothesis. The analytical method is based on the Airy's stress function method, using bipolar coordinates. The stress fields (principal shearing and principal normal stresses), plotted using Mathcad application were compared with the stress fields obtained by FEA, using Catia application. An excellent agreement is found between the plotted stresses obtained by the mentioned methods.

KEYWORDS: elastic plane, two similar holes, shear loading, stress concentration factor, Airy's stress function

1. INTRODUCTION

The paper presents analytical and FEM approaches for solving the stress state from an elastic plane with two identical circular holes. The loading of the plane at infinity is pure shearing, as seen in Figure 1. Finally, a very good agreement between the two analysis methods can be observed, despite the fact that the finite element method assumes working with finite domain and concrete elastic characteristics of the part. Muskhelishvili, [1], emphasises that for the elastic plane, under the assumptions that hole contours are free of loading, the stress state does not depend on the elastic characteristics of the material.

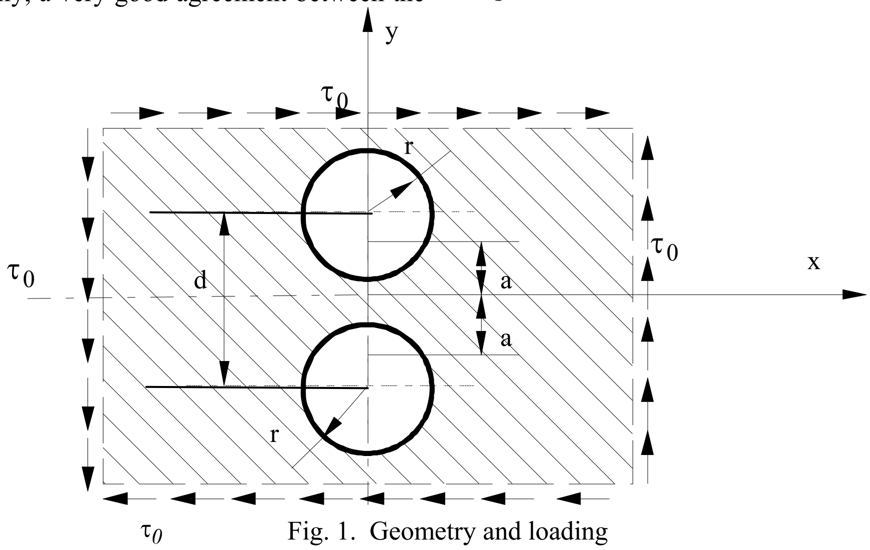


Fig. 1. Geometry and loading

2. THEORETICAL REMARKS

In order to find the analytical solution, the bipolar co-ordinates α and β are used. The mentioned co-ordinates allow the employ of simple forms of boundary conditions. The relationships between Cartesian and bipolar co-ordinates are expressed by the following equations:

$$x = a\frac{sin(\beta)}{cosh(\alpha) - cos(\beta)}, \quad y = a\frac{sinh(\alpha)}{cosh(\alpha) - cos(\beta)} \tag{1}$$

where a is a constant with length dimension. The equations for the contours of the two holes in bipolar co-ordinates are given by the relation:

$$\alpha = \pm\alpha_0 = \pm acosh\frac{d}{2r}, \tag{2}$$

where d is the distance between the holes' centres and r is the holes' radius. The "+" signum is adopted in equation (2) for the hole from the upper half-plane and the "-" signum for the hole in the lower half-plane.

The analytical solution is based on finding the Airy's function. The Airy's function is constructed in bipolar co-ordinates as a sum of two elastic potentials: first, a characteristic potential for the compact plane having the same loading and secondly, an auxiliary one, needed to impose the boundary conditions.

In Cartesian co-ordinates, the elastic potential characteristic for the sheared plane has the well-known expression, Timoshenko, [2]:

$$U(x,y) = \tau_0 xy \tag{3}$$

Jeffery, [3], shows that in bipolar co-ordinates, the auxiliary potential $U(\alpha,\beta)$ must be replaced by the function:

$$F(\alpha,\beta) = \frac{U(\alpha,\beta)}{J}, \tag{4}$$

where:

$$J = \frac{a}{cosh(\alpha) - cos(\beta)}. \tag{5}$$

Therefore, the potential function for the sheared compact plane takes the following form:

$$F(\alpha,\beta) = a\frac{sinh(\alpha)sin(\beta)}{cosh(\alpha) - cos(\beta)}\tau_0 \tag{6}$$

where τ_0 is the loading of the elastic plane at infinity. As the two holes are placed symmetrically about the axis Ox, ($\alpha = 0$), Jeffery, [3], recommends the general form of the auxiliary potential necessary to be added to the $F(\alpha,\beta)$ function, as follows:

$$\begin{aligned}\Phi(\alpha,\beta) &= \{B_0\alpha + K\,ln[ch(\alpha) - cos(\beta)]\}[ch(\alpha) - cos(\beta)] + \\ &[A_1 ch(2\alpha) + B_1 + C_1 sh(2\alpha)]cos(\beta) + [A'_1 ch(2\alpha) + C'_1 sh(2\alpha)]sin(\beta) \\ &+ \sum_{k=2}^{\infty}\left\{\begin{aligned}&[A_k ch[(k+1)\alpha] + B_k ch[(k-1)\alpha] + C_k sh[(k-1)\alpha] + D_k sh[(k-1)\alpha]]cos(k\beta) + \\ &+ [A'_k ch[(k+1)\alpha] + B'_k ch[(k-1)\alpha] + C'_k sh[(k-1)\alpha] + D'_k sh[(k-1)\alpha]]sin(k\beta)\end{aligned}\right\}\end{aligned} \tag{7}$$

The boundary conditions on the contours of the two holes are expressed by the equations:

$$\sigma_a(\alpha,\beta)\big|_{a=\pm\alpha_0} = 0 \tag{8.a}$$

$$\tau_{a\beta}(\alpha,\beta)\big|_{a=\pm\alpha_0} = 0 \tag{8.b}$$

Since the elastic plane is unlimited, a supplementary condition is required to satisfy the regularity condition for stresses at infinity, together with the boundary conditions 8.a and 8.b. It is easily noticeable that the U(x,y) potential does not produce stresses at infinity, and therefore, the above condition must be imposed only to the auxiliary potential. One can demonstrate that for an auxiliary potential tending to zero at infinity, the regularity condition for the stresses is fulfilled. The points from infinity from Cartesian plane, ($\pm\infty$), are mapped under equations (1), into the point $\alpha = 0, \beta = 0$ from the plane (α,β). As a consequence, the continuity conditions for the stresses at infinity lead to the condition:

$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}}[\Phi(\alpha,\beta)] = 0 \tag{9}$$

In order to impose the boundary conditions, the Fourier series expanding for the function $F(\alpha,\beta)$ is required. It must be emphasized that the expanding has different forms for the upper, ($\alpha > 0$), and lower, ($\alpha < 0$), half-planes, as it follows:

$$F(\alpha,\beta) = \begin{cases} 2a\tau_0\, sinh(\alpha)\sum_{k=1}^{\infty} e^{-k\alpha}\, sin(k\beta), \ \alpha > 0, \\ 2a\tau_0\, sinh(\alpha)\sum_{k=1}^{\infty} e^{k\alpha}\, sin(k\beta), \ \alpha < 0 \end{cases} \tag{10}$$

Imposing the conditions of equal coefficients for the functions $sin(k\beta)$, both from the total potential $[F(\alpha,\beta)+\Phi(\alpha,\beta)]$ and from equations for boundary conditions, 8.a and 8.b, together with (9), the unknown coefficients of auxiliary potential can be found.

The stresses are obtained in bipolar co-ordinates, using the stress function, $U(\alpha,\beta)$, via the equations:

$$\sigma_a = \frac{1}{a}\left\{[cosh(\alpha)-cos(\beta)]\frac{\partial^2 U}{\partial\beta^2} - sinh(\alpha)\frac{\partial U}{\partial\alpha} - sin(\beta)\frac{\partial U}{\partial\beta} + cosh(\alpha)U\right\},$$

$$\sigma_\beta = \frac{1}{a}\left\{[cosh(\alpha)-cos(\beta)]\frac{\partial^2 U}{\partial\alpha^2} - sinh(\alpha)\frac{\partial U}{\partial\alpha} - sin(\beta)\frac{\partial U}{\partial\beta} + cos(\beta)U\right\}, \quad (11)$$

$$\tau_{a\beta} = -\frac{1}{a}[cosh(\alpha)-cos(\beta)]\frac{\partial^2 U}{\partial\alpha\partial\beta}.$$

Another way to obtain pure shearing in an elastic plane is stretching it on one direction and compressing it on the normal direction. The distributed loadings must have the same absolute value, as shown in Figure 2.

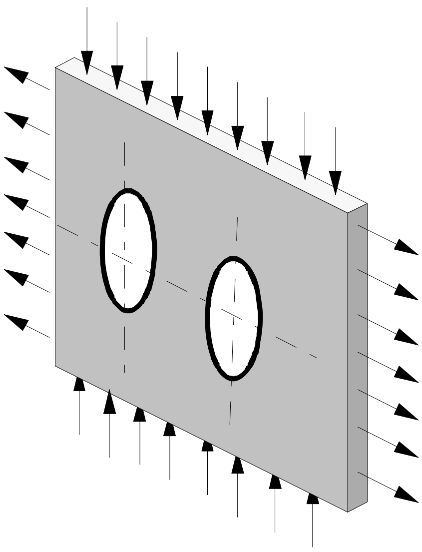


Fig. 2. The second method used for obtaining pure shearing in a plane

The stress state at infinity is characterised by the following stress tensor, expressed for the two loading methods:

$$\hat{T}_1 = \begin{bmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{bmatrix} \quad (12)$$

$$\hat{T}_2 = \begin{bmatrix} -p & 0 \\ 0 & p \end{bmatrix}$$

3. RESULTS

The stress state from a plane with two circular holes, stretched on two normal directions, one of them being parallel to the axis of the centres, was found by Alaci, [4]. Using the results and applying the superposition principle, the stress state for the second loading case can be immediately found.

In Figure 3a, there are represented the circular contours of the holes and the hoop stress variation on these contours. The polar co-ordinate representation of hoop stress variation shows only a qualitative aspect, as the shape of the curves for stress representation depends on the scale and on the considered positive sense with respect to position vector of current point on the contour of the hole originated in the centre of hole. By developing the contour of the hole, it results the Cartesian variation of the hoop stress from which quantitatively conclusions can be drawn, Figure 3b.

In Figure 4 it is revealed that the stress $\sigma_{\beta max}$ on the contour of hole presents a rapid decrease, from infinity, when the holes are tangent, to the value 3 for remote holes. The dimensionless stress was obtained by dividing the hoop stress to the shear stress τ_0 that acts on the plane at infinity. It is interesting that concentration factor has the same value as for the case of a uniaxial stretched plane with a circular hole, (Kirsch's problem, cited by Love, [6]).

The analytically isochromatics patterns, namely the principal shearing stress patterns, [5], corresponding to the above mentioned loading cases, are presented in Figure 5.

In Figure 6 are presented the maximum normal stresses, analytically found, for the two loading cases. One can observe, and the result is quite intriguing, that the stress patterns do not coincide, when the two loading cases form Figure 1 and Figure 2 respectively, are compared.

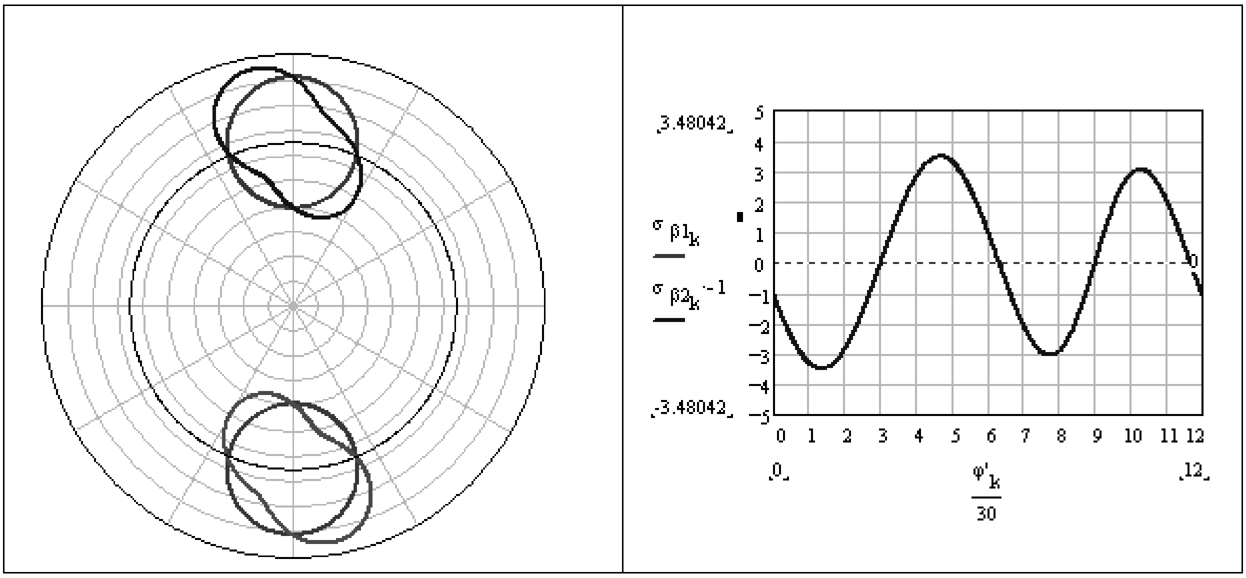


Fig. 3. Hoop stress on the holes contour

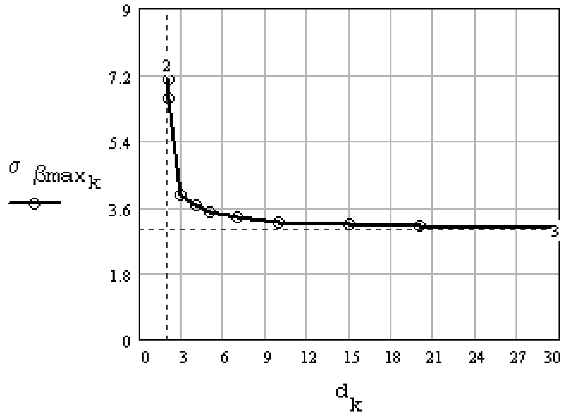


Fig. 4. Maximum hoop stress on the contour of the hole versus distance between holes centres

In Figure 6a, corresponding to loading case with shearing stresses at infinity, one can observe that the principal normal stress appears on a plane that bisects the solid angle between *Ox* and *Oy* planes. More correct, the stress tensor $\hat{T}_1$ for the case (a) is not in canonical form, as the tensor $\hat{T}_2$ is. The eigenvalues of $\hat{T}_1$ tensor are $\pm\tau_0$ but the eigenvectors are oriented at 45^o with respect to the holes' axis.

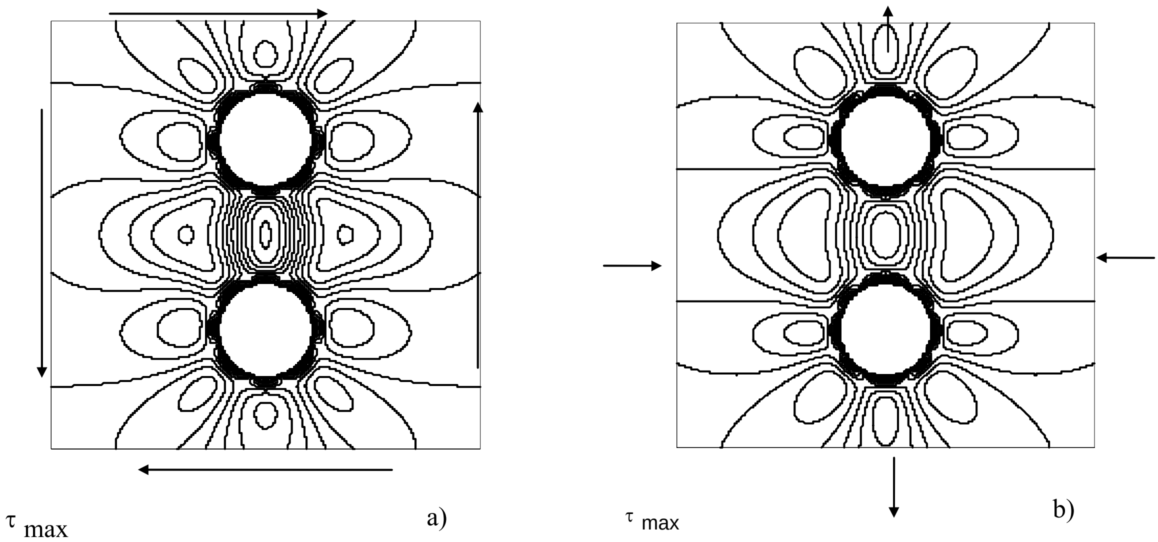
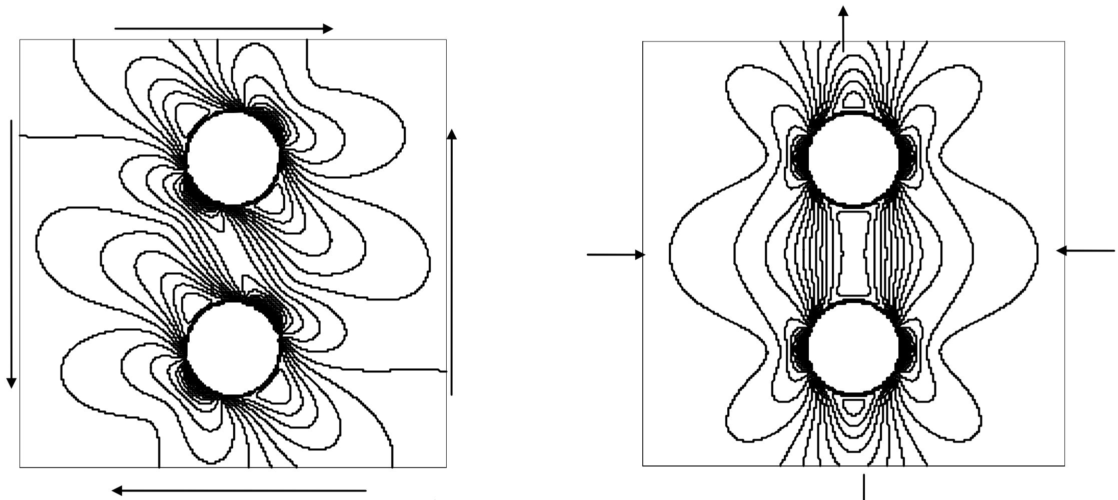


Fig. 5. Isochromatics patterns analytically obtained for the loading cases from Fig. 1, (a), and Fig. 2, (b).



σ_{max} a) σ_{max} b)

Fig. 6. Maximum normal stresses, analytically found, for the two loading cases

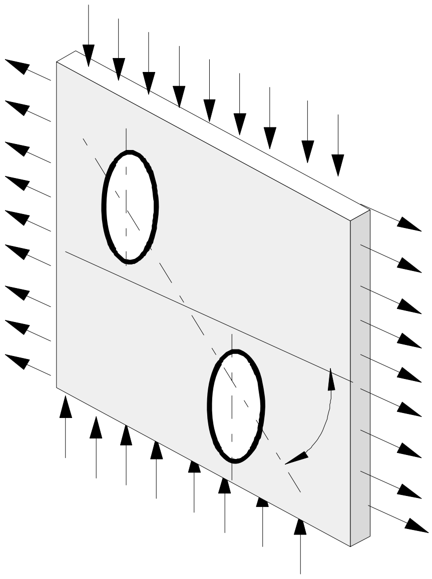
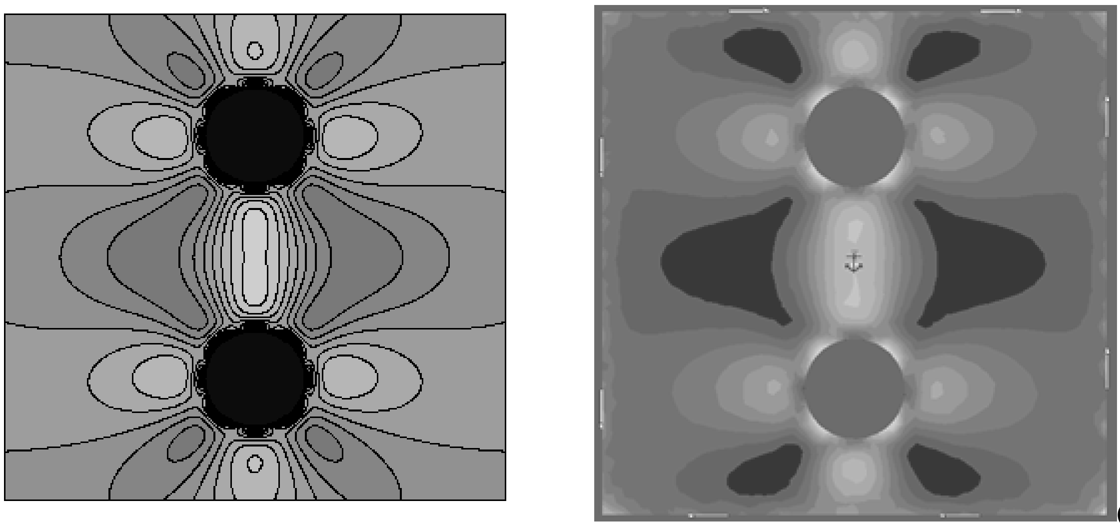


Fig. 7. Proposed loading scheme

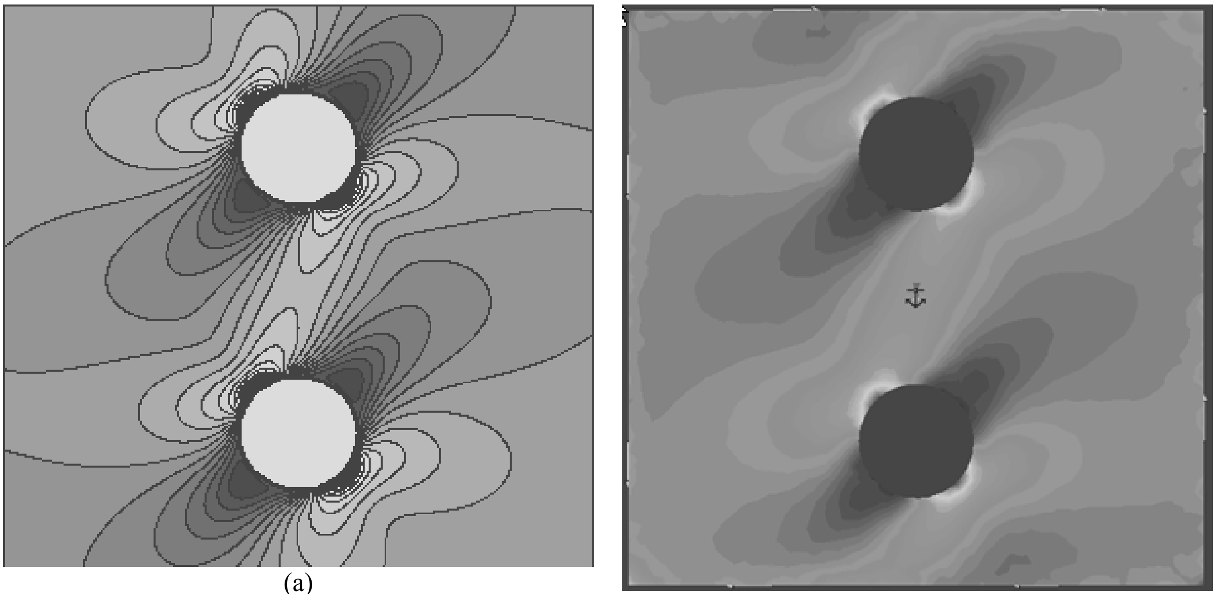
Unfortunately, the theory of elasticity gives analytical solution for a narrow number of problems. The reason is that complicated forms for the boundary conditions appear in practical applications. For the latter, currently, the most convenient method consists in numerical approach, such as finite element method. The problems having analytical solution provide as validation method for numerical procedures. In many cases it can happen that the latter have a diminished convergence or don't lead to a finite result. The loading cases presented in the paper were solved using CATIA Finite Element Analysis. The numerical results are presented comparatively with the theoretical results in Figures 8 and 9.

Comparing the graphs of the stresses obtained analytically with the stress contours obtained by FEA, one can observe a perfect agreement.



τ_{max} (a) (b)

Fig. 8. Isochromatics patterns, (a), theoretical and (b), numerically obtained by FEA



(a) (b)

Fig. 9. Principal normal stress pattern, (a), theoretical and (b), numerically obtained by FEA

4. CONCLUSIONS

The maximum stress is the hoop stress on the contour of the holes and it can be concluded that the holes are stress concentrators for the plane.

As forecasted, the concentrator effect played by the holes is strict locally, at distances relatively small from the centres of the holes (maximum 5 diameters), the stresses have the pattern similar to the stresses from a compact plane.

The maximum hoop stress appears on the contour of the holes in the points where the distance between the two contours has a minimum value.

The stress concentrator factor ranges between 4, value for the case of remote holes (there is no interaction between the holes), and any value greater than 4, as the contours come closer.

Finite element analysis is a very convenient method in evaluating the stress state for an elastic plane with different type loadings and geometries. For the present paper, the case of part with two holes, subjected to shearing, revealed stresses in very good agreement with the stresses found by the method of Airy's stress function.

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